Stein domains $E^{2n+2}$ with $\dim \leq n+1$.

First handles of $\dim \leq n$: this gives $\mathbb{F}^{2n} \times \mathbb{D}^2$ by a theorem of Cieliebak.

Take attaching spheres $A_i \subset \mathbb{C}(\mathbb{F}^{2n} \times \mathbb{D}^2)$

Suppose we can take $A_i \subset \mathbb{F}^{2n} \times \mathbb{C}^3$

Get the following picture:

![Diagram]

Remarks:

1) To build $E$, need $F_i, L_1, \ldots, L_m$ - parametrized Lagr. spheres.

2) can be extended to $\pi_E: E^{2n+2} \to \mathbb{D}^2$.

3) The symplectic orthogonal of vertical tangent space gives us a connection, $\mathcal{D}$ well defined parallel transport.

Thimbles are the points that parallel transport to the crit. point.

In favorable circumstances, can merge these two discs.
Q: Can you always have such a structure on a Stein domain?
A: Yes in dim 4, not written down in higher dimensions at all (but should be true)

Wrapped Floer homology: (wrt $n: \hat{E} \to C$ picture)

Admissible Lagrangians: let $\alpha(t)$ be vanishing paths such that

\[ \gamma(t) = t \Theta, \quad t > 0 \]
\[ \Theta \in \mathcal{S} \]

(=> rational ends)

Can define $HF^*(\Delta_0, \Delta_1)$ if $\Theta_1 \neq \Theta_2$.

\[ \Delta^\Theta = \text{rotation of the path by } \Theta \text{ at } \infty. \]

\[ HW(\Delta_0, \Delta_1) := \lim_{\Theta \to \infty} (\Delta_0^\Theta, \Delta_1^\Theta). \]

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The idea: $L$ thimble, $K$ compact exact Lagrangians

\[ [K] [L] \neq 0 \]
\[ \text{then } HW(L, K) \neq 0 \]
\[ \text{so } \text{one obstruction to the existence of such } K \]

Why? $HW(K, L)$ is a usual module over $HW(L, L)$

\[ HW(L, L) = 0 \Rightarrow HW(K, L) = 0 \]
\[ \Rightarrow HF(K, L) = 0 \]

But consider a class if this is intersection number $[K] [L]$, $\ast$

Vanishing criterion: When is $HW(L, L) = 0$? (Assume canonical bundle of $E$ is trivialized, $\mathcal{E}$ compact)

\[ HF(\Delta_0 \Delta) \to HF(\Delta, \Delta^{2\pi n^\epsilon}) \]

$\mathcal{E}$
Get a triangle

\[ \text{HF}(\Delta, \Delta) \xrightarrow{i} \text{HF}(\Delta, \Delta^{2n+1}) \]

boundary map

\[ \text{HF}(V_1, V_0) = \text{HF}(\text{sh}(V), V) \]

\[ \text{HF}(V_1, V_0) = \text{HF}(V_1 \to V_0, V) \]

\( \text{cbr global monodromy vs composition of Dehn twists?} \)

1) unit in \( \text{HF}(\Delta, \Delta) \) is the image of \( 1 \in \mathbb{Z}/2 \).

2) \( S \) can be explicitly computed.

(wrapped Floer homology vanishing \( \iff \) \( \text{not surviving} \) under \( i \)).

\( \delta \) is multiplication with some element \( \sigma \in \text{HF}^*(V_1, V, \cdots, V) \),

is in fact product of \( \sigma_k \).

\( \sigma_k \in \text{HF}^{*k}(V_1, V, \cdots, V) \), \( \text{(something)} \).

Seidel's exact triangle

\[ \text{HF}(V_1, V, V) \xrightarrow{\delta_k} \text{HF}(V_1, V) \]

Why? Map given by counting

sections

\[ L_1, L_0 \]

\( \delta = \delta_m - \sigma_1 \).

When is \( \delta = 0 \) (or \( \sigma = 0 \)?)

Seidel: \( \text{T}_{L_0} \text{L}_1 = \text{T}_{L_0} \text{L}_1 \)
\[ X_0 \to X_1 \to X_2 \to X_3 \to \cdots \]
\[ \text{Hom}(X_0, X_2) \otimes X_0 \xrightarrow{ev} X_1 \]

\[ \tau_{V_1} \tau_{V_2} \cdots \tau_{V_m} V \]

By repeatedly taking cones, we see that \( \tau_{V_1} \cdots \tau_{V_m} V \) is represented by this twisted complex:

\[
\bigoplus_{i=1}^{m} \text{CF}(V_i, V) \otimes V_i \to \vdots \to \bigoplus_{i=1}^{m} \text{CF}(V_1, V) \otimes V_1 \to V
\]

\[ \text{quotient} \]

\[ V_{m+1} \xrightarrow{\sigma} \tau_1 \cdots \tau_m V \]

\[ \sigma = 0 \Rightarrow \text{trivial splits, so } V = V_{m+1} \text{ is a summand of this quotient.} \]

\[ A_m \textbf{ singularity:} \]

\[ \mathbb{C}^{n+1} \to \mathbb{C} \]

\[ (x, z) \mapsto f(z) \to z_1^{2} + z_2^{2} + z_{n+1} \]

\[ F = f^{-1}(0) \]

\[ \tau_{v_{n+1}} \text{ projection} \]

\[ \text{clue: } V_i \text{'s are vanishing cycles for } f, \text{ i.e., take } (F, V_i, -V_m), \text{ do surgery } \to \mathbb{C}^{n+1} \text{ or } \Sigma \mathbb{V} \]
Idea: Add another path $\delta t_{m+1}$ to this collection. (In this situation, matching paths always match up to spheres)

Get $V_m \subset \mathbb{R}^{2n+2}$

For $n \geq 2$, there is only one topological/smooth $S^n \longrightarrow S^{2n+1}$.

Care can be made standard, understand framing explicitly:
surgery is $T^*S^{2n+1}$ smoothly (always if $n$ even, $[V_m]$ condition if $n$ is odd).

When is it symplectically not $T^*S^{2n+1}$?

\[ \xymatrix{ V_1 \ar[r] & \cdots \ar[r] & V_m } \]

$V_1, \ldots, V_m$ generate the following (directed) filtration category:

\[ \begin{array}{ccc}
e_1 & e_2 & \cdots & e_m \\
\delta_1 & \delta_2 & \cdots & \delta_m \\
V_1 & \cdots & V_m \\
\end{array} \]

Twisted complex over this?

\[
\otimes W_i \otimes V_i, \text{ with differential } e_i, f_i, f_{i+1},
\]

\[
(\partial_c)_{ij} = (\text{Hom } \mathbb{Z}/2 (W_i, W_j \otimes \text{Hom } (V_i, V_j)))
\]

1) General ansatz: can kill all $(\partial_c)_{ij}$ (remove unit terms)

2) Split into graded pieces, irreducible ones are:

\[
\begin{cases}
W_i = \mathbb{Z}/2 & l < i \leq k \\
\partial_i, \partial_{i-1} = 1 & l < i \leq k.
\end{cases}
\]
Geometrically corresponds to \( Y/K \), paths going from \( x^K \) to \( x^K \) end point.

Last step: Any other path cannot be a direct summand of one of these pieces.
Can easily check that any other \( Y/M_i \) can't be a direct summand,
by computing first homology of a test Lagrangian.

\[ \Rightarrow \Delta M_{x,i} \text{ has vanishing } H_0, \]

and \( \langle \Delta M_{x,i}, [0, \text{section } \mathcal{R}] \rangle \) has alg. intersection 1.

\[ \Rightarrow \text{ new } T^S^{n+1} \text{ has no compact exact Lagrangians.} \]