Janko Latschev, Symplectic homology and Lagrangian submanifolds of $\mathbb{C}^n$

1. Motivation

Fukaya's work in Lagrangians in $\mathbb{C}^n$:

$L \subset \mathbb{C}^n$ closed, oriented, rel-spin

For each $d \in H_2(\mathbb{C}^n, L)$,

$$M(d) = \{ u : (D, \partial D) \to (\mathbb{C}^n, L) | \bar{\partial} u = 0, [u] = d \} / \text{Aut}(D, \partial D)$$

that fix $L$.

**Claim 1:** $M(d)$ is a collection of chains on $\mathbb{L}$. Consider $a = \sum_d M(d) \otimes d$ on some complete chain on $\mathbb{L}$.

**Claim 1:** $a$ should satisfy some equation like

$$a + \frac{1}{2} \bar{\partial} a + \frac{1}{2} \partial a = 0$$

(maybe not strict id, maybe higher order terms in sense of $C^0$ eqn).

Next, consider the perturbed equation:

$$\bar{\partial} u = \eta_k \quad \eta_0 = 0, \quad \eta_k \to L \quad (i.e. \text{pick } \eta_k \text{ so no cohomology + eqn}$$

$$N_t(d) = \{ u | \bar{\partial} u = \eta_k \} : [u] = t^d \quad \dim N_t(d) = n + \mu(d)$$

$$N(d) = \bigcup_{t \in [0,1]} N_t(d), \quad b : = \sum_d N(d) \otimes d$$

**Claim 2:** $b$ satisfies something like $\star \star : \bar{\partial} b + \partial b = [L]$ (this generalizes Gromov's original argument)

*Note: thought of as sequence of constant loops in $L$.

we know $[L]$ is non-trivial, but if no discs, $[L] = 0$ something.
Claim 3: $HPT$:

The dg Lie structure descends as an Loo structure to $H_c(L,L)$. 

$a$ descends to a Maurer-Cartan element $a'$ here, i.e.

$$\sum_{k \geq 2} \frac{1}{k!} \bar{f}(a,\ldots,a) = 0$$

$\lambda_1 = 0$ on homology.

$b$ descends to an element $b'$ satisfying

$$\sum_{k \geq 2} \frac{1}{(k-1)!} \bar{f}(b,a,\ldots,a) = [L]$$

Note:

(generic degrees of $a, b$ are dimensions of $M, N$, degree of $f, \bar{f}$ is $1-n$)

Calculation:

- $|a| = n + 1 + m(d)$
- $|b| = n + 2 + m(d)$

for $d = 0$, this last equation says

$$\partial b + \sum_{d \in \partial \mathbb{N}} b = [L]$$

Assume $L$ is a $K(n,1)$, $H_c(L,L)$ lives in degrees $0 \leq -\leq n$. 

$\Rightarrow$ by $(\ast)$, $0 \leq n + 2 + m(d) \leq n$, $0 \leq n + 1 = q(d) \leq n$. 

(already implies $m$ can't vanish completely, needs to be positive.)

$\Rightarrow q(d) \neq 0$ for some $d$ with $m(d) = 2$ (Fukaya's theorem)

(this is oriented case, so $m$ can't be 1).

Cor: $n = 3$, $L$ closed, oriented, prime, $L \subset C^3$ Lagr. submanifold,
then $L \simeq S^1 \times \Sigma_3$, and all such actually realize by a fairly simple construction known for a long time.
Goal: Translate this into the language of symplectic homology.

If successful, we would:
- avoid chain-level string topology
- avoid relative spin assumption.
- get a generalization to other Weinstein domains.

(b/c before, we were really embedding $DT^{*}N$ into $C^{n}$)

2) Algebraic structures: (should eventually work w/o $S^{2}$-equiv.)

Prop: $W$ Liouvillian domain, $\Omega W = V$

(a) $SH^{S'}(W)$ carries an $L_{0}$ structure of degree $(2-S')$ (conjecturally same as string bracket) (coefficients are in center, so descends to $SH^{S'}$)

(b) $SH(W)$ is an $L_{0}$ module over $SH^{S'}(W)$.

(c) Any concave filling $\widetilde{W}$ of $V$
gives rise to a MC elt., $\alpha$, in $SH^{S'}(W)$ Nov = Novikov completion.

(if $\widetilde{W}$ exact, $\alpha$ will be trivial.)

Rem: together, these realize $\otimes$

(c) Bourgeois - Oancea: \[ SH^{S'}(W) = CH(W) \]
boundary operator counts formal index 1 cylinders which can consist of 2 levels,
on in $IR \times V$, one in $W$.

\[ \begin{array}{c}
1 \\
\quad
\end{array} \]
\[ IR \times V \]
\[ \cdots \]
\[ \theta_{0}, \theta_{0} \]
\[ \text{rigid-energiness planes in filling} \]

\[ \lambda \text{ counts} \]
\[ \quad \]
\[ \text{formal rational curves with k pcts. punctures and} \]
\[ \quad \]
\[ 1 \text{ neg. puncture, with 2 levels...} \]
Point: It's hard to prove geometrically that Loo relations hold. Use alg. framework of EGH to prove this. Idea is that an augmentation augments all structures that one has.

for 6. BO: $SH(W)$ can be computed from SFT pictures by using the Morse-Bott formalism.

$M$ is an Loo module over an Loo algebra $L$, $E^k \otimes \mathbb{L}^k_\mathbb{Z}$, means that there are maps

$M_k : M \otimes L^{\otimes k-1} \rightarrow M$

satisfying

$$\sum_{k_1+k_2=k}^{\pm 1} M_{k_2} (m, l_2, \ldots, l_k) \cdot l_k \rightarrow l_{k_1+k_2-1}$$

$$+ \sum_{k_1+k_2=k}^{\pm 1} \cdots M_{k_1} (m_1, l_2, l_2, \ldots, l_k) \cdot l_{k_2} \rightarrow l_{k_1+k_2-1}$$

(See 6. for $L_0$, can reorder.)
$M_k$ counts

For $M_k$:

Analyzing $d \neq 0$ is subtle b/c have all these contributions.

Funny cancellation.

(C) For the MC elf. $a$,

$\tilde{a}$ is the generating fan. Counting rigid finite energy planes is $\tilde{W}$, augmented in $W$.

Really this means things like:

all index 0
the MC equation comes from looking at how index 1 configurations of this type back
(don't want to do by hand, want to formulate algebraically)

(3) Have analogue of \( \mathcal{X} \)

Want analogue of \( \mathcal{X} \) (in prow box, we took some sort of ham. perturbation to get it)

Idea: use Gromov's graph trick to translate back into an honest hol. curve eg.

\[ S' \times L \subset C \times C^n. \]

\[ CH(T^*(S' \times L))^0 \cong H(S') \otimes CH(T^*L) \]  

up to a compact which doesn't wrap around \( S' \)

\[ CH(T^*(S' \times L))^{1,*} \cong SH(T^*L) \]

(claim: the embedding \( CH(T^*L) \hookrightarrow CH \) is an embedding of C0 algebras)

Hope: get the analogue of \( b \) by doing a deformation of the contact structure,

i.e. a deformation argument as before.

If this works, it translates to general embeddings of \( \mathcal{X} \) in Liouville domains in \( C^n \) or any other guy (we used displaceability to construct deformation)

E.g. Suppose have contact \( Y \) i.e. \( F \cup \partial \) all \( SF \) indices even

Suppose \( Y \) has stability w/ \( SF \neq 0 \). Then, this cannot have an embedding in \( C^n \)

(only interesting if not simply connected, otherwise we already knew it)

Other approach: If \( V \) CW exact embedding, have Viterbo functoriality

In the non-exact case, can do some twisting to still get a map, using fact that \( SF = 0 \).

(by these a, b).