Different parts w/ E. Gonzalez, S. Ma'ay, E. Zitter, S. Vengapalan

Setup: \( G \) compact connected, \( V \) \( G \)-module \( / C \) (\( C \)-rop in).
\( X \subset P(V) \) smooth proj. \( G \)-variety, \( X//G \) symplectic (or GIT) quotient.
Assume \( G \) acts freely.

Problem: Relate \( GW \) invariants of \( X//G \) to invariants of \( X \).

Classical story: "non-abelian localization" – take w.b., restrict to semi-stable locus, take quotient.

\[ K_G(X) \rightarrow K(X//G) \]

fails to commute by sum of fixed point contributions

\[ \text{Ind} G \quad \text{ind} \quad \mathbb{Z} \]

Can already see this when \( G = \mathbb{C}^* \), \( X = \mathbb{C}^n \), \( X//G = P^{n-1} \).

Take \( E = \mathcal{O}(d) \) \( \ell \) is ps. \# simplex

\[ \text{Ind}^G(E) = \# \{ q, r \mid \text{rdiv} \} \quad \text{Ind}(E//G) \text{ poly and} \]

Quantum story: Givental: \( X \) vector space, \( G \) torus,

\( X//G \) semi-posit. toric variety

Graph invariants for \( X//G \) related to quasimap invariants for \( X \) via

algebraically defined "mirror map"

Goal: geometric interpretation of

Gromov-Witten strategy: Gaio-Salamon, using symplectic vortices

\[ \Sigma \text{, curve } / C, \quad P \rightarrow \Sigma \quad G \text{-bundle}, \quad A(P) \text{ connections} \]

\[ \text{a gaged holomorphic map } = \{ \varphi(A, u) : A \in A(P), u : \Sigma \rightarrow P \times_G X, \text{ s.t. } \int \varphi_{A u} = 0 \} = \mathcal{H}(P, X) \]

\( s(P) \text{ gauge transformations} \)

makes sense b/c already have herm. vert. c.s.

A gauge splitting, so \( s < \text{ total c.s.} \)
\[ \text{Vol}_\Sigma \in \mathcal{L}^2(\Sigma) \text{ area form } g = g^x. \Phi : X \to g^x \]

\[ P(\Sigma) : P \times g^x \to P \times g^x \]

\[ \text{Moment map } \]

\[ \mathcal{H}(P, X) \to \mathcal{L}^2(\Sigma, P \times g^x) \]

\[ M(P, X) = \mathcal{H}(P, X) / g(P) \text{ moduli of symplectic vectors} \]

\[ M(\Sigma, X) = \bigcup_{\text{types } P} M(P, X) \]

\[ \overline{\mathcal{M}}_n(\Sigma, X) \text{ versi"on } \}

\[ \text{ev} \]

\[ f \]

\[ X^G \]

\[ \overline{\mathcal{M}}_n(\Sigma) \text{ stable maps} \]

Baez construction:

gauged GW invariant:

\[ H_G(X)^{W_n} \otimes H(\overline{\mathcal{M}}_n(\Sigma)) \to \mathbb{Q} \]

defined in good cases by Solomon & collaborators.

Mandet's thesis \[ \overline{\mathcal{M}}_n(\Sigma, X) = \left\{ \begin{array}{c} \text{hol. } G \text{ on } P \cr \text{sectors } \Sigma \to P \times g^x \cr \text{with stability condition} \end{array} \right\} \]

* Gonzalez - W invariant for proj. \[ X \].

Dependence on \[ \text{Vol}_\Sigma \] :

Gaiotto - Solomon \[ \text{Vol}_\Sigma \to \infty \].

Unfortunately occur generally in general.

\[ \overline{\mathcal{M}}_n(\Sigma, X) \to \overline{\mathcal{M}}_n(\Sigma, X / G) \text{ + bubbles} \]

good cases \[ \text{formula for GW of } X / G \].

Gonzalez - W \[ \text{Vol}_\Sigma \to 0, \Sigma = \mathbb{P}^1 \]

In no canonical symplectic str.,

Canonical degenerate symplectic str.

(goodness still make sense)
wall crossing formula for variation in $\text{Vol}_X$.

Idea: incorporate bubbling into alg. str.

three types of bubbles:
- sphere bubbles in $X/G$ (not $P^1$ but
  - vertices on $C$, $dx dy$ of different
  - sphere bubbles in $X$

Ziltener:

\[
\text{compactification of}
\]

\[
\overline{M}_n(C, X), \text{ of vertices on } C \text{ w/ n-markings away from } X
\]

Feu"e,

\[
\overline{M}_n(C)
\]

Case when $X = \mathbb{C}$ = p.t.

Defn. J. Wehrheim: quasimaps $\sim$ symplectic vertices if $X$ vector space $G$ forms $\triangleright$

Want: $H^*_G(X) \otimes \otimes H(\overline{M}_n(C)) \to H(X/G)$

Venugopalan $\triangleright$ Hitchin-Kobayashi correspondence for vertices on $C$

\[
(\text{case } X = C, G = C^{\times})
\]

\[
\text{in Jaffe-Taubes}
\]

\[
\text{Proof uses gradient flow of } \lVert F_A + u^* P(\overline{\mathbb{C}}) \rVert_{L^2}
\]

Alg. interpretation as morphism of CohFT’s.

Defn: $\text{A}_X \otimes \text{CohFT}$ consists of a vector space $V$ and a collection of maps

\[
\mu^n : V^\otimes n \otimes H(\overline{M}_{0, n+1}) \to V
\]

satisfying a splitting axiom.

Example: $V = \otimes H^*_G(X) = H_G(X) \otimes \Lambda^* X$

$\mu^n$ defined by pull push over $\overline{M}_{0, n+1}(X)$

(giving up inner product b/c it’s not preserved by morphisms).

(Actually using CohFT, using inner product to move one factor over, forget inner prod.).
Let \( V, W \) be \( \text{Comm}_\mathbb{C} \) algebras.

A morphism of \( \text{Comm}_\mathbb{C} \) algebras from \( V \) to \( W \) is a collection
\[
\phi^n : V \otimes H(M_{\alpha/n}(\mathbb{C})) \to W \tag{\text{splitting axiom}}
\]

(\text{i.e. a morphism of Coh FTS is one of Comm \( \mathbb{C} \) alg. forgetting their product.)

\[
M_{\alpha/n}(\mathbb{C}) = \mathbb{C}^n
\]

complexifications of Stasheff's multiplication.

just as \( \overline{M}_{\alpha/n} \) complexified associahedra.

Thus: Under same technical hypotheses, diagram of Coh FTS / traces
\[
GW_G(X) \xrightarrow{QK} GW(X/G)
\]

quantum Kirwan morphism

\[
\int. \text{ over } M_\alpha(\Sigma X/G)
\]

\[
\int. \text{ over } \overline{M}_\alpha(\Sigma X/G)
\]

stable maps to \( \Sigma X/G \)

degree \( 1 \) on \( \Sigma \).

Fails to commute by wall crossing terms

(What has changed since 2 years ago are the technical hypotheses)

(Exh: like a "trace" but in a different Novikov ring from you began with).

Question: Relate to mirror symmetry?

\( X \) vector space, \( G \) torus.
\[
GW_G(X) \xrightarrow{QK} GW_G(X/G)
\]

\[
\Delta_X \xrightarrow{\text{graph}} \Delta_X
\]

\[
\Delta_X \xrightarrow{\text{counts}} J \text{ function}
\]

\[
\text{Gromov's language}
\]

\( F \) function
Potential functor:  

\[ \text{Conn alg} \rightarrow F \text{ manifolds} \quad \text{Mannin} \]

\[ V, \mathbb{G}^n \quad \text{morphism} \quad \rightarrow \quad V, \text{Fun}(V) = \sum \frac{1}{n!} \pi_n(v, \ldots, v; 1) \]

paritals define \( \partial_v \) on \( T_v V \)

\[ \phi \quad \rightarrow \quad \phi : V \rightarrow W \quad \phi \quad \rightarrow \quad \sum \frac{1}{n!} \phi_n(v, \ldots, v; 1) \]

\[ D_v \phi : T_v V \rightarrow T_{\phi(v)} W \quad \text{is a homomorphism} \]

\[ \text{* potential } \quad \Phi : \quad F_{\text{grass}} = F_{\text{graph}} \circ QK \]

\[ \quad \text{(equivalent to work of Catanese, others, I & J relations)} \]

\[ \cdot \text{ twistings seem ok.} \]

Remarks:  

(i) no restrictions on \( C_1 \), \( G \) non-abelian possibly.

(ii) \( T \triangleleft G \) max torus

\[ F_{X/G} = F_{X/L} \]

(formula \( F_{X/L} \circ QK \) twisting \( = F_{X/L} \circ QK_G \))

(iii) \( QK \) homomorphism

\[ Q \Phi_G(x), \mathbb{C}^n \rightarrow Q \Phi_C(X/L) \quad QK(v) \]

Note \( QK(0) \neq 0 \) in general. (analogue of \( u^0 \) for \( \text{A} \))

up curvature of morphism

relations by looking at Ker \( DQK \).

(iv) behaviour of \( L \) of \( X/L \) under birational equivalence of git type.
Example of (iii): $X = \mathbb{C}^r$, $G = \mathbb{C}^*$. $X/G = \mathbb{P}^{r-1}$.

$QH_G(X) = Q[x, z]$. $Q_K(0) = 0$.

$QK^* : h^0$. $QH_G(X) \to QH(X/G) = Q[x, z]/K^0$. $QK^*$.

Coefficient of $QK(x^*) = \int eV_{\infty} x^* \left( \prod_{i=1}^{2r} eV_{\infty} x^* \right)$

$\mathcal{M}_1(G, X) = \{ (a_1, z^{-b_1}, \ldots, a_r, z^{-b_r}) \} + \text{bubble. eV}_{\text{start}} [a, \ldots, a_r]$

$(a_i, -b_i) \neq 0$

$eV_{\infty} x^* = \{ (z^{-b_1}, -b_2, \ldots, -b_r) \} \cong \mathbb{C}^r$,

but $X = \text{Eul}(G, \mathbb{C})$, $x^* = \text{Eul}(\mathbb{C}^r)$ (Un $x = \mathbb{C}^r$)

coeff = # zeros of any section = $\# eV^{-1}(0) = 1$.

$QH(\mathbb{P}^{r-1}) = Q[x, z]/x^r = Q$. Similar for any toric (not necessarily Fan).

Batyrev, Givental $q > 2$ $\Rightarrow$ not new.

McDuff - Tolman - Iriani

[Khao Nguyen: description of $H^2$].