Tobias Ekholm, Legendrian Contact Homology and symplectic homology in dimension 4

Two joint projects:
- Joint w/ Bourgeois-Elashvili
- Joint w/ Murphy-Ng

Surgery formulas for Fisker homologies

Legendrian homology in \( \#(S^2 \# S^2) \) (empty connect sum is \( S^3 \))

Examples and applications.

Setup:

\[ \omega = d\lambda \quad (Y \times \mathbb{R}, \lambda = e^{t\alpha}) \]

\[ d\lambda_{|X} = 2\alpha \]

\[ H: X \to \mathbb{R} \text{ Morse Gen.} \]

Critical points have index \( \leq n \).

(By Gielisch, if no index \( \geq n \) at crit. points, then \( X = X \times \mathbb{C} \) symplectically).

\[ \text{Ex: dim } X = 4 \]

Symplectic invariants:

(Assume \( c_1(X) = 0 \) for talk)

- Contact homology:
  \[ R \text{ Reeb v.f. on } Y, \quad d\lambda(R, \cdot) = 0, \quad \alpha(R) = 1 \]
  \[ P(Y) = \{ \text{proper orbits} \} \]
  \[ \text{CH}(X) = \mathbb{Q} < P_{\text{good}}(Y) > \]

Fix an acs \( J \) on \( X \) compatible w/ \( \omega \).

\[ d_{\text{CH}}(Y) = \sum_{\beta \in \{ \beta: |\beta| = 1 \}} K_{\beta} \left( \left[ \left( \beta \right| \text{in } Y \times \mathbb{R} \right) \right] \beta \]

(Pink: Alternatively, could use \( \frac{1}{K_{\beta}} \) here).

Explanation of dots in \( \mathbb{N} \): Actually, our picture looks like:

\[ Y \times \mathbb{R} \]

\[ \text{Then: } d_{\text{CH}}^2 = 0, \]

\[ \text{CH}(X) = H_0(\text{CH}(X), d_{\text{CH}}) \]

bad orbits: multiples, s.t. the parity of the CH index is different from the underlying simple one.
Symplectic homology (as defined by Banyaga–Ozora)

\[ \text{Symplectic homology} \]

\[ \mathcal{S}H^+(X) = \mathbb{Q} \langle p(y) \rangle \oplus \mathbb{Q} \langle p(y) \rangle [1] \]

Geometric interpretation:

\[ \begin{array}{c}
\text{max} \\
\text{min}
\end{array} \]

\[ d_{\text{Morse}} \delta = \begin{cases} 
2 \delta & \text{if } \gamma \text{ is bad} \\
0 & \text{if } \gamma \text{ is good}
\end{cases} \]

\[ \Theta(\gamma) = \sum_{|\beta| = 2} \# \left( \begin{array}{c}
\gamma \\
\beta
\end{array} \right) \delta \]

Two more pieces to differential/p ordinary Morse differential, and

\[ \mathcal{Q}(p(y)) \to \text{Morse } (-H) \]

\[ d_{\text{Morse}} (\gamma) = \sum_{|\beta| = 0} \# \left( \begin{array}{c}
\gamma \\
\beta
\end{array} \right) \delta \]

Example:

\[ \mathbb{R}^2 = \mathbb{R}^2 - \{a_1, a_2, \ldots, a_n\} \]

In the limit, all Reeb chords become very long except \( y_1 \).

So \( \text{CH}^+ \) is done.

\[ \text{SH}^+ \]

\[ y_1, y_2, \ldots, y_n \]

So \( \text{SH}^+ \) has rank 1.
For $SH$, so $SH = 0$.

In fact, with a more subtle argument, for any smooth manifold, $SH = 0$, $SH^+ = H^+_o$ (or $H^+_2$), $CH =$ some equivariant version.

Relative counterparts:

$CL = X$

$H^+ = \{ \alpha \in C^*(\Gamma) \}$

$LH(\Gamma) = Q < C(\Gamma) >$

$\partial_{LH}(c) = \sum \pm \left( \begin{array}{c}
\partial_c \in (Y \times \mathbb{R}, \Gamma \times \mathbb{R}) \\
b \end{array} \right) b$

What do these really look like?

$\Lambda \subset Y$ Legendrian submanifold

$LHA(\Lambda) = T(LH(\Lambda))$

$LCH(\Lambda) = \ldots$

Attach an $n$-handle:

$CH(Y)$

$LCH(\Lambda) = CH(Y_0) \oplus LH_{\text{cyc}}(\Lambda)$,

where

$LH_{\text{cyc}}(\Lambda) = LHA(\Lambda) / \sim$

where $c_1 \sim \cdots \sim c_m \sim (-1)^{\ell c_2 + \ell c_m}$

$w \rightarrow (w)$
If $l$ is odd, then $(a^3) = 0$.

Differential on $LH^{cyc}(\Lambda)$ is induced by $d_{LHA}$.

$L_{SH}(X) = SH(X_0) \oplus LH^0(\Lambda)$.

So to calculate $SH$, it's enough to know

Handle: c co core sphere

Then $LH(c) \cong LHA(\Lambda)$.

Example:

One can prove that it suffices to stay in $\mathbb{R}^3$ to compute differential (ellipsoid / long Reeb chord argument again)

$X^2$ projection

Chee handles are attached, Kirby diagram.
\[ C(A) = \begin{cases} \text{double points in diagram, chords inside handle 3} \\
\text{n-strands through handle} \\
x_{ij}^0 & 1 \leq i < j \leq n \\
x_{ij}^k & 1 < i, j \leq n, \ k \geq 1. \end{cases} \]

\[ \partial_n x_{ij}^0 = \sum_{m=1}^{n} x_{im}^0 x_{mj}^0 \]

(convention that \( x_{ij}^0 = 0 \) if \( i \geq j \))

\[ \partial_n x_{ij}^1 = \delta_{ij} + \sum_{m=1}^{n} x_{im}^0 x_{mj}^1 + \sum_{m=1}^{n} x_{im}^1 x_{mj}^0 \]

\[ \partial_n (x_{ij}^k) = \sum_{l=0}^{k} \sum_{m=1}^{n} x_{im}^l x_{mj}^{k-l} \]

Integral difference
Outo differential:
Count polygons like:

Test cases:

$T \times T^2$:

([Diagrams of topological structures with labels and arrows indicating attaching 1-spheres])

Exotic $\mathbb{R}^6$ (after some doubling?)
Can get an exotic $\mathbb{R}^6$ w/ this technique.