Talk 1: Paolo Rossi, Hamiltonian Systems and Topological Recursion in SFT.
(joint w/ O. Fabert)

Cylindrical case: $\mathcal{M}^A_{g, r, n^+, n^-}$, $A \in \mathbb{H}_2(V)$

$F^{\pm} = \varphi_{\gamma_1}, \ldots, \gamma_{\pm} \in \mathfrak{g}$ two sets of Reeb orbit

Want to compactify: $\overline{\mathcal{M}}^A_{g, r, n^+, n^-}$

codimension 1: multi-floor curves

nodal curves

codimension 2

Gravitational Descendants: Tautological bundles

$\overline{\mathcal{M}}_{r+1} = \overline{\mathcal{M}}^A_{g, r+1, n^+, n^-}$

$\pi: \overline{\mathcal{M}}_{r+1} \to \Sigma$

$\pi_{r+1}$

$\tilde{\Sigma}_{r+1}$ = cotangent sheaf to the fibre (not a bundle b/c we have an orbifold)

$\chi_{i, r} = \sigma_i^*(\tilde{\Sigma}_{r+1})$ bundle (marked pts. stay away from nodes)
**Comment** γ-classes

\[ M \times M^* = \mathcal{M}^* \rightarrow M^* \]

Here, * is the i-th marked point.

Choose connectors on \( \mathcal{L}_i \) coherently on the boundary

\[ \psi_{i,r} = C_{2i}(\mathcal{L}_i, r) = \text{curvature}. \]

**Comment** Sections: Choose sections \( \Delta_{i,r} \in \mathcal{T}(\mathcal{L}_i, r) \) coherently on the boundary

\[ M \subset M(u, k) = \left( \Delta_{i,r}^+(0) \cap \Delta_{i,r}^-(0) \right)^{k-1}_{\text{sections} \ s_{i,r}^0 \ \text{of} \ \pi(\mathcal{L}_i, r)}. \]

**SFT Potential:**

\[ H = \sum_g H_g \ t^g \]

\[ H_g = \sum \int_{M_{g,0}^{k_1, k_r}} \text{ev}_i^*(\Theta_{k_1}) \cap \ldots \cap \text{ev}_r^*(\Theta_{k_r}) \]

where \( M_{g,0}^{k_1, k_r} = \cap M(i, k_i) \)

Unlike GW case, \( H \) actually depends on the differential forms \( \Theta_{k_i} \), not just their cohomology classes.

\[ H(t, p, g, z, h) \quad (\Theta's \ are \ representatives \ of \ \Theta_{k_i}). \]
Thm (EGH): $[H_1, H] = 0$, where we define $[,]$ as:

$$([P_\theta, 2_\gamma] = u_\gamma i)$$

Multiplicity of Reeb orbit.

Invariance: $W$, $d_{H_1} := [H_\gamma, \cdot]$, $(d_{H_1}^2 = 0$ by above theorem).

(EGH)

Consider $[H_1] \in H_x(W, d_{H_1})$.

(Proof: If we changed the contact form, $W$ would change, but the same result is true).

For different choices $H^+, H^-$, we have:

$$\psi : H_x(W, d_{H^+}) \rightarrow H_x(W, d_{H^-}).$$

(and it's known how this isomorphism behaves). Under above,

$$[H_1^+] \xrightarrow{\psi} [H_1^-]$$

(As one might imagine, this map is gotten through a cobordism).

Hamiltonian: $H_\alpha \dot{e} := \frac{\partial H}{\partial x} \dot{e} \Rightarrow [H_\alpha \dot{e}, H_{1_\beta}] = [0]$.

$$[H_{1_\alpha}] \xrightarrow{\psi} [H_{1_\beta}]$$

Geometry of descendants:

$$\overline{M}_r \leftarrow \overline{L}_i \xrightarrow{\pi}_r$$

$$\overline{M}_{r-1} \leftrightarrow \overline{L}_{i, r-1}$$

$\pi^{-1}_r (\overline{M}_{r-1}) \cup D_{i, r} = \overline{M}_r$ (really, you can make a coherent choice of sections so that this holds: non-unique choice).
In $\mathbb{H}_+(W, dh)$: $t = \sum t^{k+1} \Theta_\alpha$

\[
\frac{\partial H}{\partial t^0} = \sum t^{k+1} \frac{\partial H}{\partial t^0} + \int_{V} \angle t \angle t.
\]

\[
\frac{\partial H}{\partial t^0} = \sum t^{k+1} C_{\beta} \frac{\partial H}{\partial t^0} + \angle \angle (\angle \angle, H)
\]

This holds only up to homology.

\[\frac{\partial}{\partial t^0} H = D_{Euler} H \quad (D_{Euler} = 2 t \frac{\partial}{\partial t^0} + p \frac{\partial}{\partial p} + q \frac{\partial}{\partial q} + t \frac{\partial}{\partial t^0})\]

Ranks the Euler characteristic of the curve.

This is implicitly summing over all $p, q, \alpha, \beta$.

For any $p, q$, $p \frac{\partial}{\partial p} + q \frac{\partial}{\partial q}$ has degree 0.

Idea of proof: Choose coherent sections that ensure (*).

With this assumption, the proof becomes similar to GW theory.

In order to understand degree of curve, need cohomology surfaces.

May be more difficult if $\gamma$ is not compactible.

But make some fixed choices.

$2$ factor actually depends on these choices.

This gives the form $(A, H)$ above, which is functorially exact.

This sum $D_{Euler} H$ is covariant w.r.t. Euler characteristic is additive w.r.t. gluing of curves.

(Remark: $D_{Euler}$ doesn't have much to do w/ Euler v.s. in the Frobenius manifold)
TRR (Topological Recursion Relation) Some sort of reasoning as above, but
(still work in progress) were subtle.

Every time you have.

\[ \begin{array}{c}
\text{glue in SFT} \\
\text{asymptotic} \\
\text{only remembers orbit.}
\end{array} \ \\
\hline
\begin{array}{c}
\text{but in TRR, seem to want to} \\
\text{remember the relative } \theta \text{ angle} \\
\text{in the gluing parameter.}
\end{array}
\]

As such, ends up being not quite a recursion.

When

\[ S' \text{- symmetry in } V \text{ is the same as } \theta \text{ parameter, can find } \theta \text{ parameter} \]

for \( \theta \) some \( S' \) parameter at marked point.