Seidel: Lagrangian spheres \( S \cong S^n, \quad S \in \mathbb{C}^n, \quad [S] \in H_n(M), \quad (n \geq 1). \)

Since \( S \) is \( n \)-dimensional, \( [S] = (-1)^{\frac{n(n+1)}{2}} \chi(S) = \begin{cases} 1 & \text{if } n \text{ even} \vspace{1em} \cr 0 & \text{if } n \text{ odd} \end{cases} \)

due to homology

- \text{even: } \Rightarrow [S] \text{ is non-zero and primitive in } H_n(M)/\text{torsion}.

Moreover, if \( S_i, \cdots, S_k \) are pairwise disjoint, then \( [S_i] \) are linearly independent.

For \( n \) odd: \( ? \) topological intersection doesn't say much.

Example: The flag manifold \( M = FL(C^3) \) contains a Lagrangian \( S^3 \) but \( H_3(M) = 0. \)
(Remark: Floer says doesn't see this as a sphere, which is part of the problem.)

Look instead in a Lefschetz manifold: any 3-manifold has Lagrangian spheres? Still unknown.

Some local results (surprisingly):

- Theorem: Take \( M = T^* S^n \) then \( [S] = \pm [S^n] \), and moreover any two such \( S \) must intersect.

- Theorem: Take \( M = \text{Milnor fibre of an } (A_n) \) singularity. (Linear plumbing of \( n \) spheres).

Then, \( [S] \) is non-zero and primitive. Moreover, if \( S_i, \cdots, S_k \) are pairwise disjoint, the classes \( [S_i] \in H_n(M, \mathbb{Z}/2) \) must be pairwise distinct.

(Remark: for \( n = 2 \), Aboveon-Smith; much stronger than this result. In fact, a much more general thing is true: topic for today!)

(Theorem 4) Let \( M \) be a Lefschetz manifold, \( c_1(M) = 0 \). Suppose that \( M \) admits a dilatation.

Then, there is an upper bound \( N \) on the size of collections of disjoint Lagrangian spheres.

(Theorem A) Let \( \pi : E \to D \) be an exact Lefschetz fibration, \( c_1(E) = 0 \). Let \( N \) be the fibration, \( (S_i, \cdots, S_k) \) a basis of vanishing cycles, and let \( B \subseteq \text{Fuk}(M) \)

be the associated full sub-category.
Suppose that $\mathbb{B}$ has a $C^*$-action, which is dilating with weight $1$.

Then, if $S \subset E$ is a Lagrangian sphere,

$$[S] \in H_n(E, [\mathbb{M}]) \cong \mathbb{Z}$$

is non-zero and primitive.

Moreover, if the homology classes of disjoint Lagrangian spheres are linearly independent.

Ex. $p \in C[x_1, \ldots, x_{n+1}]$ polynomial, $p(0) = 0$, and $0$ is an isolated critical point.

The Milnor fibre is

$$\{ x \in C^{n+1} \mid p(x) = \epsilon, ||x|| < \delta \}$$

Choose $0 < |\epsilon| \ll \delta \ll 1$.

Corollary: Suppose that

$$\phi(x) = x_1^2 + x_2^2 + \cdots + x_{n+1}^2.$$ 

(Am: special case).

Then Theorem A applies with $E$ being the Milnor fibre.

Ex. ($M = \mathbb{T}^n \times L(1, 2) )^{-2}$ has sphere

has a dilatation. Therefore, the resulting bundle is $N \leq \mathbb{Q}_2$ (linear in $\epsilon$).

(Problem: there aren't any Lagrangian spheres in $M_0 + \epsilon$ all!)

Note: All the results apply to real homology spheres which are spin or, more generally, spherical objects of the Fukaya category. Moreover, disjointness can be replaced by vanishing of floor cohomology.

Taking the 0-section with different flat line bundles gives $\mathbb{Q}$ spherical objects,

so the band (1) isn't that far.

(Rem: not clear that classical homology is the right place for these bands).

[5 - Solomon]

$C^*$ action, dilating of weight $1$ - purely geometric condition, not quite real, but similar ideas.
Definition: Let $B$ be a proper $A$-module category over $C$. A $C^*$-action on $B$ is called dilating at weight $d$ if there is a quasi-iso morphism of equivalent bounded modules

$$B \cong B^{
u \Phi} [-n] \otimes V_d$$

where a $C^*$-action is fixed on objects, acts a morphism in a way compatible with composition.

Examples: $M = T^* L$, $L$ smooth, simply-connected ($\Rightarrow$ smooth), and formal.

Then, $\text{Fuk}(M)$ has a $C^*$-action which is dilating, and weight $d = n$.

Noting:

$$\text{Fuk}(M)^{tw} \cong C^+ (L)^{tw} \cong H^+ (L)^{tw}$$

(up to equivalence, $H^+$ (Abelian), formal).

$C^*$ acts on $H^+ (L)$ acts with weight $i$ in degree $i$ (Fubini-Study field).

(Principle: Formality is symmetry, higher $A$-structures break actuations).

Corollary: Suppose $B$ has a dilating $C^*$-action of weight $d$. Let $X$ be an equivalent twisted complex which is spherical.

$$H^+ (\text{hom}(X, X)) = \bigoplus_{x = 0}^{C^* \times = 0} H^x$$

Then, the action of $C^*$ on $H^+ (\text{hom})$ has weight $0$ in degree $0$.

Sketch of proof of Theorem A: $\tau : E \to D$, fibre $M$.

$$\leq \text{Fuk}(M)$$

full subcategory with objects is basis of vanishing cycles. (see $\rightarrow S_k$).

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Then, \( \text{K}_0(E) \hookrightarrow A^+ \)

\[ \text{K}_0(\text{K}_0(E)) \rightarrow \text{K}_0(A) \]

\[ \downarrow \]

\[ \text{H}_n(E) \rightarrow \text{H}_n(E, \mathbb{Z}) \cong \mathbb{Z}^k \]

Under the assumption of Theorem A,

\[ \text{K}_0^\times(A) \cong \mathbb{Z}[t, t^{-1}]^k \]

\[ \downarrow \]

\[ \text{K}_0(A) \rightarrow \mathbb{Z}^k \]

("Klein pairing").

\[ \text{K}_0^\times(A) \] has a pairing with values in \( \mathbb{Z}[t, t^{-1}] \), written as \((-,-)^\times\), such that

\[ ([\alpha], [\beta])^\times = (t \mapsto \text{Str}(t : H^\bullet(\text{Hom}(\alpha, \beta)))) \]

Suppose we have \( \alpha \in \text{K}_0(A) \), satisfying \( (\alpha, \alpha)^\times = 1 - t \).

(We take from hypotheses: \( \text{Str} \) of weight -1 \( C^\times \) acts on spaces, b/c dual n has weight 1.)

Then, the range of \( \alpha \) in \( \text{K}_0(A) \) is non-trivial.

Otherwise, \( \alpha = (-1)y \Rightarrow (\alpha, \alpha)^\times = (-1)(-1) \cdot (y, y)^\times \) doesn't divide.

(Follows that reduction to \( \text{K}_0(A) \) is non-trivial.)

Q: How did you make \( \alpha \) \( C^\times \)-equivalent? Ans. in general, \( C^\times \) acts most non-trivially.

Case 1: If base object is 0, define it, so non-relevant in hom, then it's always equivalent. (Pull back by \( C^\times \)-modules, integrate.)
Claim: if you stabilize a Calabi-Yau fiber 4 times, you actually get $C^4$-ack.

**Dilate**:

- $M$ (an orbifold manifold), $c_1(M) = 0$, $t_B$ operator
- A dilation is a class $B \in SH^2(M)$ satisfying $\Delta B = 1 \in SH^0(M)$.

**Example**: $M = T^*F$, $F$ flag variety in $C^n$ admits a dilation.

- $M = T^*S^n$, $n > 1$ admits a dilation.

never quite revisited:

- (?) The Milnor fiber of any singularity $p(x) = x_1^2 + x_2^2 + x_3^2 + g(x_4, \ldots)$ admits a dilation.

- (?) Cotangent bundles of formal manifolds? E.g. fiber over $T^*S^2$ pull up dilate from $T^*S^2$.

(n-1 family of loops which each have c/ degree 1.

- e.g. great circle loop around $S^n$ from $L$ every 2 get cell, give dilate in $HF^*(S^n) \simeq HF^*(S^2)$ closed.

1 If $M$ has adiabatic $L_0, L_1 \subseteq M$ are Lagrangian submanifolds with $H^2(L_0) = 0$ and which are spin, then

$$\Phi_{L_0, L_1} : HF^*(L_0, L_1) \to HF^*(L_0, L_1)$$

is canonical up to adding constants

[get eigenspace decomposition], and such that

$$\Phi_{L_0, L_1} = \Phi_{L_1, L_0}$$

is the identity (requires $\Delta B = 1$). (So it's non-trivial).

We can define $L_0 \cdot \cdot L_1 = Str (\Phi_{L_0, L_1}) = \sum x_{\lambda}^x \chi_{\lambda}^x$ for $\lambda \in \mathbb{C}$, $\sum \chi_{\lambda}^x \in \mathbb{Z}$, $\mathbb{Z}$ Edu. clew of eigenspace of $\Phi_{L_0, L_1}$.

and then for a Lagrangian $S^1$, $

S^1 \cdot S^1 = 1 + (1)^n$.
This comes from

\[ SH^*(M) \to HH^*(\text{Fuk}(M)) \]

send \( B \) to infinitesimal action on \( \text{Fuk}(M) \).

Whether you can exponentiate to an actual \( C^* \) action, no one knows.

This is the mirror of \( C^* \) actions on the B-side.

Rumb: there ought to be a smooth, non-poor analogue. You should be able to prove that if \( \Delta B = 1 \), then

\[ [B, C_Y] = C_Y \]

where \( C_Y \in HH^0(M, M) \).

(Do should imply weaker equivalence)