Def: \( A_\mathbb{Q} \) algebra (over field \( k \))

- \( \mathbb{Z} \)-graded vector space \( A \)
- multilinear maps \( m_n : A^\otimes n \to A \quad (n \geq 1) \).
  \( \deg(m_n) = 2 - n \)
- Satisfying:
\[
\sum_{i+j+l = n} (-1)^{ijl} m_{i+1+2} \left( 1 \otimes m_j \otimes l \otimes 1 \right) = 0 \]

sign convention: \((f \circ g)(x \otimes y) := (-1)^{|f||g|} f(x) \otimes g(y)\).

\( n = 1: \quad m_1^2 = 0 \Rightarrow \text{call } m_1 = d. \quad (A, d) = \text{chain complex} \)

\( n = 2: \quad m_1 \circ m_2 - m_2(1 \otimes m_1) - m_2(m_1 \otimes 1) = 0. \)

Def \( x \cdot y := m_2(x \otimes y). \) Then, this is saying:
\[
\Rightarrow \delta(x \cdot y) = dx \cdot y + (-1)^{|x|} x \cdot dy \quad , \text{i.e. } m_2 \text{ well-defined on homology}.
\]

\( n = 3: \quad x \cdot (y \cdot z) - (x \cdot y) \cdot z = d m_3(x \otimes y \otimes z) + m_3(dx \otimes y \otimes z) + (-1)^{|x|} m_3(x \otimes dy \otimes z) + (-1)^{|x|} m_3(x \otimes y \otimes dz) \)

\( \Rightarrow m_2 \text{ associative on } H^* A. \)
Examples: (Special cases)
- \( A^p = 0 \ \forall \ p \neq 0 \Rightarrow \text{assoc. algebra} \)
- \( m_n > 0 \ \forall \ n \geq 3 \Rightarrow \text{differential graded (dg) algebra} \)

From now on, work over \( \mathbb{Z}/2 \) & ignore signs!!

Cohomology picture:
- \( \Rightarrow \) all \( m_n \) have deg = 1.

Define:
\[
D : A[[1]]^* \rightarrow \overline{T A[[1]]}^* = \bigoplus_{n \geq 1} A[[1]]^* \otimes_n \]
\[
D := \sum_{n \geq 1} m_n^*.
\]

Extend by Leibniz rule to \( D : \overline{T A[[1]]}^* \rightarrow \overline{T A[[1]]}^* \)

Then, \( m_n \) define an \( A_{dg} \)-alg structure \( \iff D^2 = 0 \).

Pictorially:
\[
D = m_1^* + \bigoplus y m_2^* + \bigoplus y m_3^* + \cdots
\]
\[
D^2 = 1 + y + y + y + \cdots = 0.
\]
\[ D^2 = 0 \iff \sum_{m_i} = 0 \]

Yanki: homology of total \( D \) is Hochschild homology??

E.g. Chekanov homology

\[ L = \text{Legendrian knot} \]

\[ A = \mathbb{Z}_2 < a_1, a_2, a_3, a_4, a_5 > \quad a_i \text{ are crossings.} \]

Label regions of plane by \( \frac{+1}{-1} \)

\[ \text{contributes a term } b_1 b_2 b_3 b_4 \]

\[ \text{to } D a_i \]

\[ \text{e.g.} \]

\[ D a_1 = a_3 a_4 a_5 + a_5 + a_3 + 1 \]

Get a curved or weak \( A_{\infty} \) algebra (has \( m_0 \) term).

Thm: (Chekanov): \( D^2 = 0 \), homology is indep. of leg. isotopy

Can you argument this to get an honest \( A_{\infty} \) alg. (simpler invariant).

Why is this called curved?
A structure arising from a vector bundle:

\[ E, \nabla, A = \Omega^2 (\text{End}(\xi)) \]

\[ \nabla \big|_M = \nabla_0, \quad m_1 = d \nabla, \quad m_2 = \wedge^2 m_{11} \wedge, \quad \]

\[ m_{n=3} = 0 \]

\[ \text{Aoo-equations } \Rightarrow \text{Blanchi, curv. def.} \]

(Yank: \( D^2 = 0 \) is regular Maurer-Cartan eqn).

Morphisms of Aoo-algebras:

**Def:** A morphism of Aoo-algebras should correspond to a chain map in coderivation picture.

\[ F \text{ consists of } f_n : A^{\otimes n} \to B \text{ s.t.} \]

\[ \sum f_n (\mathbf{1}^{\otimes n} \otimes m_3^A \otimes \mathbf{1}^{\otimes n}) = \sum m_r^B (f_i; \alpha - \partial f_i) \]

\[ F^{DA} = D^{BF} \iff \sum \psi \psi = \sum \psi \psi. \]

\( n=1: f_1 m_1 = m_1 f_1 \Rightarrow f_1 = \text{morphism of ch. cpx.} \)

\( n=2: f_1 m_2 = m_2 (f_1 \circ f_1) + m_1 f_2 + f_2 (m_1 \otimes 1 + 1 \otimes m_1) \)

\[ \Rightarrow f_1 (x, y) = f_1 (x) - f_1 (y) = f_2 (x, y) + f_2 (\partial x, y) + f_2 (\partial y, x) \]

\[ \Rightarrow f_1 : \mathbb{H}^* A \to \mathbb{H}^* B \text{ is an alg. hom.} \quad f_2 (x, dy) \]
Morphisms are composed in the natural way:

\[(f + Y + \cdots) \circ (f + Y + \cdots) = f + Y + Y + \cdots\]

Morphisms \(f, g : A \to B\) are \(A_\infty\)-homotopic if the maps \(\overline{T}A[I]_* \to \overline{T}B[I]_*\) are chain homotopic.

**Def:** An \(A_\infty\) morphism is a quasi-isomorphism if \(f\) induces an iso. on homology.

**Thm:** Given a quasi-iso. \(F : A \to B\), there exist \(G : B \to A\) \(A_\infty\)-quasi-iso. which is a homotopy inverse of \(F\).

**Perturbation Lemma:**

\(A = \mathcal{X}_\infty\) alg.

\((B, d_B) = \text{chain complex}\)

\(f : (A, m_A^1) \to (B, d_B)\) chain map

\(g : (B, d_B) \to (A, m_A^1)\) homotopy inverse of \(f\).

i.e. \(1 - gf = dh + hd\).

Then there exist, canonically, \(\epsilon\):
1) Ano structure on $\mathcal{B}$ extending $d_B$

2) Ano-morphism $F : A \to \mathcal{B}$ extending $f$.

3) " " $G : \mathcal{B} \to A$ " " $g$.

4) Ano-homotopy $H$ between $GF, I$, extending $h$.

Proof: Define $D^3_B : g$

$$D^3_B = d^3 + Y + \cdots + Y$$

(internal edges labeled w/ $h$).

$(D^3_B)^2 = 0$?

All directed trees w/ one outgoing edge $\rightarrow$ planar $\Rightarrow$ 2 meaning at each vertex

No $\deg 1$ internal edges?

$$\sum \text{all possible trees from bottom to top,}$$

$$\text{internal edges labeled } h, \text{ vertices labeled } w_i, A.$$

(one term for this) is:

\[ \text{planar} \]
\[ D^3 = 1 \cdot d^3 + \sum_f \]

\[ (D^3)^2 = \]

\[ \Rightarrow \]

\[ \Rightarrow \]

\[ \Rightarrow \]

\[ \Rightarrow \]

\[ (\text{using } 1 - g f = d h + h d) : \]

\[ \Rightarrow \]

\[ \Rightarrow \]

\[ \Rightarrow \]

\[ \Rightarrow \]

\[ \Rightarrow \]

\[ \Rightarrow \]

\[ = \]

\[ = \]

\[ = \]

\[ = \]

\[ = \]

\[ (d^2)^2 = 0 \]
Do the same for $G, H \to$ slightly hairier.

**Cor:** There is a minimal $(m_i=0)$ $A_{\infty}$-alg structure on $H^*A$ if $A$ is $A_{\infty}$-alg.

**Proof:** Choose a "Hodge decomposition":

$$A = B \oplus C \oplus D,$$

and define:

$$f: A \to B \text{ projection}, \quad \text{inclusion} \quad D \to A$$

$$g: B \to A \text{ inclusion}$$

$$h: A \to A, \quad h = 2g \circ \left( d \circ 2g \right)^{-1} \circ \pi_c$$

and $1-gf = dh + hd \text{ (exercise)}$. (Also, need to show $f g = d^B f = 0, \quad d g = g d^B = 0$)

so that we have chain maps.

$\Rightarrow H^*A$ has $A_{\infty}$-alg. structure with $m_i = 0$, and $E_{\infty}$-morphisms $A \cong H^*A$ quasi-isomorphisms, $G \cong \sim 1$.

Proof of them that quasi-isos are invertible:

$$A \xrightarrow{F} B$$

$$\text{H}^*A \xrightarrow{H=\text{gofoi}} \text{H}^*\beta$$
Lem: If \( \tilde{F}: \tilde{A} \to \tilde{B} \) is an \( \text{endo morphism} \), 
and \( \tilde{A}, \tilde{B} \) minimal (\( m_\tilde{A} = m_\tilde{B} = 0 \)),
then \( \tilde{F} \) admits an inverse \( \tilde{G} \) such that \( \tilde{F} \circ \tilde{G} = 1 \) and \( \tilde{G} \circ \tilde{F} = 1 \).

(Actual inverse, not up to homotopy.)

Quick idea of proof: Can just look at relations to explicitly determine \( \tilde{G} \):

\[
\tilde{F} = 1 + \psi
\]

\[
\tilde{G} = 1 + \psi
\]

\[
\begin{array}{ll}
\text{Case } n=1: & \\
\text{Case } n=2: & \\
\end{array}
\]

\[
\begin{array}{l}
A \xrightarrow{F} B \\
H^* A \xrightarrow{h^{-1}} H^* B
\end{array}
\]
Let \( G = i \cdot H^{-1/2} \).

Then \( FG = F \cdot i \cdot H^{-1/2} \)

\[ \sim j q \cdot F i p i H^{-1/2} \]

\[ = j H p i H^{-1/2} \]

\[ \sim j H H^{-1/2} \]

\[ \sim j q \sim 1. \]

Remark: Construction of Aus structure on \( H^*A \) depends on choice of "Hodge decomposition" so can try to do this on a Kähler manifold using actual Hodge decomposition.

Definition: \( A \) is formal if there is an \( A_{\infty} \) quasi-isomorphism \( F: A \rightarrow B \), where \( B \) is minimal \( \& \ M^B_{n > 0} = 0. \)

Then (Merkulov): \((M, \omega)\) Kähler \( \Rightarrow \Omega^*(M, R) \) formal. (idea: \( A = \Omega^*(M, R) \), \( B = \ker d^* \) of \( \Omega^d \), i.e. \( A_{\infty} \) alg. \( B \sim B \sim H^*A \).