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Symplectic and Contact Homology for Symplectic Fillings

Ref: Bourgeois - Oancea: An Exact Sequence for Contact 3-Manifolds

\((W, \omega)\) symplectic manifold s.t. \(\partial W = M\) contact

\(\exists X \in W \text{ s.t. } X \text{ is outward pointing along } M\)

Then \((\mathcal{W}, \omega)\) is a symplectic filling of \((M, \alpha = \iota_X \omega)\).

\(\mathcal{L}_X \omega = \omega ?? \text{ at least near } M ??\)

Linearized contact homology

\[\begin{align*}
1 \times \mathbb{R} &\twoheadrightarrow \mathfrak{g} \times \mathbb{R}, \text{ so need to allow for differentials}\ \\
\text{the filling allows us to define an augmentation } \epsilon : \{\text{Reeb orbits}\} &\rightarrow \wedge \\
\gamma &\rightarrow \# \{\text{Reeb orbit}\} \rightarrow \parallel \bigwedge \langle \gamma \rangle
\end{align*}\]

\[\begin{align*}
\hat{W} &= W \cup M \times [0, \infty) \\
\hat{\omega} &= \hat{\iota}_X \omega \text{ on } \hat{W} \\
\text{d(e}^t \gamma) &\text{ on } M \times [0, \infty)
\end{align*}\]

\[C_\ast(\mathcal{W}) = \bigoplus \Delta \langle \gamma \rangle\]

contact form \(d: C_\ast(\mathcal{W}) \rightarrow C_{\ast+1}(\mathcal{W})\)

\[L \text{ C H}(\mathcal{W}) = H_\ast(C_\ast(\mathcal{W}), d)\]

graded by \(\text{deg} (\gamma) = M(\gamma) + (n-3)\).

\[\text{molar } / \text{CZ index.}\]

\[\text{Actually, } \text{e}(\xi_1) \text{e}(\xi_2) \chi\]

\(\text{Ricci quotient, by } S^2 \text{ symmetry in domain.}\)
Need to break $S^1$-symmetry (e.g., be able to change Hamiltonian, break $S^1$-symmetry).

**Parametrized LCH:**

Want: consider CH for parametrized Reeb orbits.

Generic case: $A(t)$ time dependent, $t \in S^1$.

Instead, note that for a constant (as usual), Reeb orbits are in $S^1$-families. This corresponds to the "Morse-Bott case."

**Morse-Bott homology:**

- Let $M$ be (another) manifold, $f : M \to \mathbb{R}$ Morse-Bott function, $\text{Crit}(f) = \bigcup C_i$ submanifolds (of non-degenerate hypersurfaces to crit.

For any $i$, pick a Morse function $g_i$ on $C_i$ (synonym: Morse-Smale conditions).

$c^0$ perturb $f$ by $g_i$ to Morse function $f_i$.

- Back to contact case: $f_0$.

By analogy, pick Morse functions (with only one max & min) $f_t$ on the $S^1$-families $S_t$ of (parametrize) Reeb orbits.

Now, define chain complex

$$\text{PC}_* (\mathbb{R}) = \bigoplus_{\text{Reeb}} \Lambda^{\langle \gamma_{\text{min}}, \gamma_{\text{max}} \rangle}$$

differential sends cascade of cols. over connected by trajectory.
\[
\begin{align*}
\text{(1)}: & \quad \text{Lemma (1): Given } H = (H, \cdot) \\
\text{(2)}: & \quad \text{Lemma (2): Given } H = (H, \cdot) \\
\text{(3)}: & \quad \text{Lemma (3): Given } H = (H, \cdot) \\
\text{(4)}: & \quad \text{Lemma (4): Given } H = (H, \cdot) \\
\text{(5)}: & \quad \text{Lemma (5): Given } H = (H, \cdot) \\
\end{align*}
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