1993 M.K. "homological algebra of mirror symmetry"

1992 Fukaya Aoo-categories from sympl. field $\mathcal{M}$
der. category from alg. field

Vague idea: sympl. geom. is broader than alg. geom.
namely noncomm. derived space over non-archimedean field $\mathbb{C}(t)$

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Derived noncomm. alg. geom. (A: Bondal, 80s)

$X$ scheme $/k \to$ small tri. cat. $\text{Perf}(X)$ perfect complexes

$\text{Perf}(X) \subset D(Q\text{Coh}(X))$ complexes which are locally (in Zariski top) quasi-isom to finite length complexes of fg. proj. modules

Conversely: interpret any small kahabi-closed triangulated cat.
as $\text{Perf}$ of some "space."

In this world, "everything is affine."

Thm (Bondal–Van der Bergh)

\begin{align*}
\forall X \text{ finite type separated scheme } / k, \exists \text{ dg. algebra } A \text{ (not unique)} \\
\text{ s.t. } \text{Perf}(X) = \text{Perf}(A-\text{mod})
\end{align*}

i.e. direct summands of $M$, $M$ free finite $R_T$-graded $A$-module

generators $m_1, \ldots, m_n$, $\deg(m_i) \in \mathbb{Z}$, $d m_i \in A m_1 + \ldots + A m_i$ (strictly triangular diff).

\begin{itemize}
  \item If $X$ is smooth & proper
    \begin{align*}
    \Rightarrow & \text{ A has additional properties: } \\
    & \left\{ \begin{array}{l}
    A \in \text{Perf}(A \otimes A^{op} \text{-mod}) \\
    (\text{Say A smooth, proper if these hold}) \\
    \dim H^*(A) < \infty
    \end{array} \right.
    \end{align*}
  \end{itemize}

\begin{itemize}
  \item If $X$ is Calabi-Yau $K_X = 0$ $\Rightarrow$ smooth, or proper w/ Gorenstein sing.
  \end{itemize}

\begin{itemize}
  \item If $X$ is smooth, say $A$ is CY if $R\text{Hom}_{A-\text{mod}-A}(A, A \otimes A^{op}) \simeq A[N]$

  \item For proper $A$, say $A$ is CY if $A^e = \text{Hom}(A, k) \simeq A[N]$

The 2 definitions agree if $A$ is smooth & proper.
Smoothness notion isn't good enough in this world.

Pathological example. \( A = k[x_i, \frac{1}{x_i-x_j} \mid i \in S] \) \( S \) infinite is smooth!!

Better notion: Toën-Vaquier 2005

\[ A \text{ is of finite type}, \text{i.e. } A = \text{homotopy retract of some } A', \]
\[ A' = k \langle x_1, \ldots, x_N \rangle, \text{deg } x_i \in \mathbb{Z}, \text{dx}_i \in k \langle x_1, \ldots, x_{i-1} \rangle \]
(can denote such \( A \) by a finite quiver, \( x_i \) = arrows)

Finite type \( \Rightarrow \) smooth, and \( \Leftrightarrow \) for compact \( A \).

What about non-smooth spaces? Get a nonsmooth space \( X \) as follows:

\( X \) smooth, no cusp, space \( \Rightarrow \) closed subset

\[ \text{mean } \mathcal{Z} = \text{supp}(M), \text{ME Perf}(X) \]

\[ \text{Perf}(Y) = \text{Perf}_{\text{supp } \mathcal{Z}}(X) \cdot \text{Perf}(B\text{-mod}), \text{ } B = \text{End}_A(M)^{op} \]

Namely: completion of \( \mathcal{Z} \); we kill \( M \) by localization.

ie: equivalently: adjoin \( \epsilon_M, \text{deg } \epsilon_M = -1, d\epsilon_M = id_M \).

Formal neighborhood: \( \text{Perf}(C\text{-mod}), \text{ } C = \text{End}_B(M)^{op} \)

\( \text{Ex: } A = k[x] \text{ affine line}, M = k \) \( (x = 0) \)

\( \Rightarrow B = H^0(S^1, k), \text{ } C = k[[x]] \).

Finite type nc spaces: examples:

1. \( \text{Perf}(X) \text{ } X \text{ smooth finite type commut. scheme} \)

2. \( \text{Perf}(A\text{-mod}), \text{ } A = \text{Chains}, (\Omega(X, x_0)) \text{ based loop space} \)
   \( \text{of } X \text{ finite CW complex} \text{ } (\mathbb{Z}_{\geq 0} \text{-graded}) \)
wlog assume $X$ is simplicial subcomplex $\Delta^n$

$\Rightarrow$ dg-quiver $Q_X$: $\text{Ver}(Q) = X_0$, vertices of $X$

for $k \geq 1$, arrows of quiver:

$0 \leq i_0 < i_1 < \ldots < i_k \leq N$ s.t. span$(i_0, \ldots, i_k)$ is a $k$-face of $X$,

define arrow $\alpha_{i_0, i_1, \ldots, i_k}: i_0 \to i_k$ of deg $= 1 - k$

with $d\alpha_{i_0, i_1, \ldots, i_k} = \sum_{j=1}^{k-1} (-1)^j \alpha_{i_0, i_1, \ldots, i_j \cdot i_k}$

Consider Path algebra $\text{Path}(Q)$ -- dg-algebra

+ need to invert $\alpha_{i_0, i_1}$'s as follows to get $A$

Rmk: $E \xrightarrow{\alpha} F$ or invertible $\iff\text{Cone}(E \xrightarrow{\alpha} F) = 0$

ie $E[1] \otimes F$ needs to carry a modified differential

$\Rightarrow$ need to introduce 4 arrows: deg $h_{E^X}$, $h_{F^X} = -1$

$\Rightarrow$ deg $h_E = -2$, $h_{E^X} = 0$

$d h_{E^X} = \text{id}_{E^X} - h_{F^X} \circ$

$d h_{F^X} = \text{id}_{F^X} - \circ h_{E^X}$

$d h_{E^X} = 0$

$d h_{F^X} = \circ h_{E^X} - h_{F^X} \circ$ give homotopy

ie. $h_{F^X}$ is homotopy inverse of $\circ$

other give the homotopies.

$\Rightarrow$ to invert $\alpha_{i_0, i_1}$'s, add 4 arrows for each of them,!!

NB: if $X = K(\Gamma, 1)$ then can take $A = \mathbb{Z}[[\Gamma]]$

$\text{Perf}(A\text{-mod}) \supset \text{f.d. dim. } A\text{-mod}$

$= \text{f.d. dim. reps of } \Gamma$. 
(3) $S$ separated scheme of finite type

$\Rightarrow D^b \text{Coh}(S) = \text{Perf}(A_{\text{mod}})$, $A$ finite type

($\text{Perf}$ in formal neighborhood $x$ in $Y$ ....).

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**Derived nc scheme from symplectic geometry:**

- $Y$ compact symplectic manifold $\Rightarrow L$ compact singular Lagrangian subvariety.

- $A_L$ finite type dg-algebra ($\mathbb{Z}/2$-graded), Calabi-Yau, depending only on a formal neighborhood of $L$.

- For smooth $L$ this agrees with $C_0(L)$ (example (2)).

- Holomorphic counts in $Y$ give a solution

  $\gamma = \sum \gamma_i T^{e_i}, \ e_i > 0, \ e_i \to +\infty$

  of Maurer-Cartan eqn.

  $d\gamma + \frac{1}{2} [\gamma, \gamma] = 0$

  in Hochschild complex $C^\bullet(A_L, A_L) = \text{Cone}(A_L \to \text{Der}(A_L))$

  $A_\infty$-deformations and gauge transf's

**Conditions:**

- $\text{Perf}(A_L)$ = homotopy colimit of certain

  constructible sheaf of finite type cat's over $L$

**Elementary building blocks:** = cat's of $\mathbb{Z}/2$-graded reps of $A_n$-quiver

$1 \to 2 \to \cdots \to n$

$\text{Aut} \{(\text{Perf}_{\mathbb{Z}/2}(A_n-\text{mod})) = \begin{cases} \mathbb{Z}/2 & n=1 \\ \mathbb{Z}/n \ltimes \mathbb{Z}/2 & n \geq 2 \end{cases}$
Cyclic symmetry \((\mathbb{Z}/n\mathbb{Z})\) is between (e.g. \(n=4\))

\[
\begin{align*}
\text{Ext}^1 & \rightarrow (0100) & \text{Ext}^1 \\
(1000) & \rightarrow (0010) \\
\text{Ext}^0 & \leftarrow (1111) & \text{Ext}^0 \\
& \leftarrow (0001)
\end{align*}
\]

\*

**Example:**

\(\text{Nbd}(L) = \text{Stein mild}\)

\(L = \text{Large skeleton of Stein mild}\)

should have sing. \~ "soap bubble" sing.

Ambient symplectic structure \Rightarrow cyclic ordering of edges

On a Lagrangian with soap-bubble singularities, put a constructible sheaf of categories:

\(\rightarrow\) on smooth points, \(A_1\)

(or, if orientation reversed, opposite = \(\mathbb{Z}/2\) shift)

\(\rightarrow\) on codim 1, \[
\begin{align*}
\text{codim 1,} & \quad A_2
\end{align*}
\]

with reduction functors to \(A_i\)’s for the 3 edges

\(\rightarrow\) etc. codim \(k \rightarrow A_{k+1}\).

Giving things together is not completely obvious:

example:
quiver has 2 vertices for each edge

edge \( \xymatrix{v & x \ar[r] & x' \ar[r] & v'} \) homotopy inverse

(\* homotopy \* add 4 arrows \*)

vertex \( \xymatrix{v & x \ar[r] & x' \ar[r] & v'} \) \( \text{Ext}^1 \)

\[ \text{Ext}^1 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \]

\[ \text{Ext}^1 \left< \text{Ext}^1 \right> \subseteq \text{Cone}(\mathcal{E} \rightarrow \mathcal{F}) \]

ie. make \( \text{Cone}(\mathcal{E} \rightarrow \text{Cone}(\mathcal{E} \rightarrow \mathcal{F})) = 0 \)

by adding 12 arrows \!!!

So... get a large dg-quiver, continuously defined

(without using any particular coefficient ring !)

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Example: Large skeleton \( \text{Perf}(A_2) \approx D^b \text{Ch}(X) \text{ alg. geom.} \)

* point \( \xymatrix{\mathbb{R}^n} \) \( \Rightarrow \) point

\( \text{Perf}(A_2 \text{-mod}) = D(\text{Vect}) = D^b(\text{Ch}(\mathbb{P}^1)) \)

* \( \xymatrix{S^1} \) \( \rightarrow \) \( \mathbb{A}^1 \setminus \{0\} \)

* \( \xymatrix{\mathcal{A}_2} \) \( \rightarrow \) \( \mathbb{A}^1 \)

* \( \xymatrix{\mathbb{P}^1} \) \( \rightarrow \)

* \( \xymatrix{\mathbb{A}^1 \cup \mathbb{A}^1} \)
Examples suggested:

- $\text{Aut} \, (\text{limiting Fukaya cat. of nbd of } L) \cong \pi_0 \left(\text{Symplectomorphisms of nbd of } L\right)$
  
  - Does this make sense after deformation by holom. disc? 

  - $G_0: \text{Aut} \, D^b(k3 \text{ surface}/\mathbb{C}(t)) = \pi_1 \left(\text{moduli space of mirror dual}\right)$

Stability for Fukaya categories:

- For 2nd kind, $\mathcal{N} \in \mathcal{N}(Y) \otimes \mathbb{C}, \det = 0$
  
  - $GL_+(2, \mathbb{R})$ acts ($\mathbb{C} = \mathbb{R}^2$)

  - Traditionally require $\mathcal{N}$ holom. wrt $C^*$. Structure - better:
    
    - More generally, $\forall \mathbb{R}^{n-1} \subset T_y \mathcal{Y}$ isotropic $(n-1)$-plane, spanning $v_1 \ldots v_{n-1}$

    - $\mathbb{R}^2 = (\mathbb{R}^{n-1})^\perp / \mathbb{R}^{n-1} \rightarrow \mathbb{C}$

    - $u \mapsto \langle ? , u, v_1 \ldots v_{n-1} \rangle$ has det > 0.
(on B-side: Shab \iff ample class, Kähler form)

Indeed: \{ e^{x^n} + \cdots \ dx \} = \text{Shab}(A_n\text{-mod})

1-form on $\mathbb{C}$

(check by C. Fabre-Kor)

\[ \text{Shab} \mathcal{D}^b(\mathbb{C}^n) \text{ has } \text{Sp}(2n,\mathbb{Z}) \times \text{GL}_+ (2,\mathbb{R}) \text{ action} \]

\[ \text{generic elliptic curve} \quad \Leftrightarrow \quad \pi_0 \text{Symp}(T^{2n}) \text{ minor} + \text{modif. } C \cong \mathbb{R}^2 \]