Ponitz, HMS 8 (closed-open string maps)

with N. Sheridan

Phase 2: also w/ S. Garbars

Setup: \( \text{Aside} \quad (X^n, \omega) \quad \text{quasi-cyclic!} \)

- C (TX) formalized
- \( D \subset X \) codim-\( 2 \) symp! submfd
  (or more generally s.n.c. divisor)
- \( \Theta \in \Omega^2(X, D) \) w/ \( d \Theta = \omega \big|_{X \setminus D} \).

We'll be using the relative Fukaya category

\( F(X, D) \): curved \( \mathcal{A} \mathcal{C} \) cat. over \( K = \mathcal{C}(q) \).

\( \text{Ob:} \) cop/ exact kernels \( \in X \setminus D \).

\( \text{Hom:} \) usual \( F \) for co-chain spaces, \( c \in \mathcal{K} \).

\( \text{mor:} \) \( \exists d \geq 0 \) involve pseudohol. polygons \( \gamma \) regular by \( x, u \).

\( \Phi = \text{tw} F(X, D)_{\mathcal{A} \mathcal{C}} \) split-closed triangulated \( \mathcal{A} \mathcal{C} \) category.

B-side: \( \mathcal{X}_\mathcal{A} \) cx. analytic family, smooth, proper, rel. dim \( n \).

\[ \downarrow \]

\( \Delta^* \subset C \)

\( \mathfrak{g} \quad \mathfrak{g} \quad \mathfrak{g} \quad \text{ample rel. bsc.} \) (give \( \mathfrak{g} \) to \( \mathfrak{g} \))

\( \mathfrak{g} \) holo. n-fan, non-deg. on fibres.
\[ \Rightarrow \text{alg. var.} \quad X \quad \text{"Laurant expansion of } \mathbb{X}_n \text{"} \]

\[ \text{Spec } \mathbb{K} \quad \text{from a projective embedding } \mathbb{X}_n \text{ in } \mathbb{P}^N(\mathbb{K}) \]

Imagine:

\[ X \neq \mathbb{X}_n \text{ mirror pair, certified by some sort of T-duality.} \]

Large error limit assumption on \( \mathbb{X}_n \):

\[ \text{monodromy } T : \text{Aut}(\mathbb{H}(\mathbb{X}_n, Q)) \text{ is maximally unipotent.} \]

\[ \text{e.g. } (T-I)^{n+1} = 0 \quad \& \quad (T-I)^n \neq 0. \]

(mirror + \([w] = 0\), so better have this anyway).

Hypothesis: "core HMS"

In this setup, say we're given full sub-categories \( A \subset F \)

\[ B \subset \text{perf } \mathbb{X}_n \text{ perfect derived category (dg model), s.t. } B \text{ split-generates perf } \mathbb{X}_n. \]

and a quasi-equivalence \( A \simeq B \).

Qe: approachable any trivial method? known only for a few cases so far)

Generation thesis: (P.-Shender):

Core HMS \( \Rightarrow \) \( A \) split-generates \( F \), hence \( F \) is (homologically) semistable,

\[ \& \text{ moreover } F \simeq \text{perf } \mathbb{X}_n. \text{ (HMS holds).} \]

Isomorphism theorem (P.S.):

Core HMS \( \Rightarrow \) open/closed maps \( HH^b(F) \xrightarrow{\text{op}} QH^b(X) \xrightarrow{\text{co}} HH^b(F) \).

\( \text{are isomorphisms.} \)
8. preface, \[ \int_X [D]^n = \int_{\bar{\mathcal{Z}}} (\nabla_{\bar{\partial}})^n \bar{\mathcal{Z}}. \]

In quintic compact cases, e.g.,

\[ \text{a SM in } \mathcal{Z}, \text{ (case of small cases)} \]

where \( \bar{\mathcal{Z}} \) is normalized so that Floer-Pompe duality \( A \leftrightarrow \text{Fun} \) care thus
duality in \( \mathcal{Z} \) w.r.t. \( \bar{\mathcal{Z}} \)-homology of canonical bd.

"Phase 2" (w. Shenker & Gaitsie):

Explains the role of Gauss-Manin connections \( \text{VHS} \).

Use that to show that core \( \text{HMS} \Rightarrow \bar{\mathcal{Z}} \) is Hodge-theoretically
normalized, & \( \mathcal{Z} \) is a canonical coordinate.

\[
\begin{align*}
\text{Spec} H_n^\text{DR}(\mathcal{Z}) &= W_{2n} \Rightarrow \cdots \Rightarrow W_0 > 0, \\
\text{monodromy weight filtration}
\end{align*}
\]

\[
\text{want:} \ \ \nabla_{\partial} \bar{\mathcal{Z}} \in W_{2n-2}, \text{ (normalization), } \& \\
\left( \frac{\partial}{\partial \bar{\partial}} \right)^2 \bar{\mathcal{Z}} \in W_{2n-4}, \text{ (canonical coordinate).}
\]

Expected consequence:

Shenker proved \( \text{HMS} \) for quintic 3-fold, up to an undetermined mirror map,

\[ \text{this then } \Rightarrow \text{ the mirror map is fixed} \]

\[ \bar{\mathcal{Z}} \text{ is standard normalized} \]

* Condolos et al. come count follows: (too computing those numbers!)
Closed string data:

1. \( H^0 \) — graded until \( K \)-algebra.
2. \( H_0 \) — """" \( H^0 \)-module.
3. Pairing \( H_0 \otimes H_0 \rightarrow K \)
4. "marking" \( K \in H^2 \).

morphisms of such data:

\[ H^0 \xrightarrow{f^*} K^0 \quad \text{homomorphism, resp. marking} \]
\[ H_0 \xrightarrow{f_0} K_0 \quad \text{isomorph. w.r.t. pairing.} \]

Ex: (i) \( H^0 = H^0 T(U) = \bigoplus_{p+q=0} H^p(\Lambda^q T\mathcal{X}), \)
\[ H_0 = H_0 \Omega^1(U) = \bigoplus_{p-q=0} H^p(\Lambda^q T\mathcal{X}). \]

Pairing: \( \langle a, b \rangle = \int_{\mathcal{X}} ab. \) "may be non-symmetric?"

Marking \( \Theta \) is the Kodaira-Spencer class in \( \frac{d}{dz} \in H^2(T\mathcal{X}) \in H^2. \)

(ii) \((\mathcal{A}, J)\) is a proper A∞ category, i.e. \( H^2(\text{hom}(X, X)) \) finite dim.

\( \Theta \) closus this data from Hochschild invariants:

\[ H^0 = H^0 \mathcal{A}, \quad \text{Marking } K = \left[ \frac{d}{dz} \right], \quad \text{easier to define but result independent of choice.} \]
\[ H_0 = H_1 \mathcal{A}, \quad \text{Nakai pairing} \]
\[ H^2 \mathcal{A}. \]
a quasi-equipped $A_2 \rightarrow \mathcal{A}_2 \rightarrow \text{isomorph. closed string data,}$

Modified, global $H/K/R$ isomorphisms give an isomorphism

\[
H^*(X) \cong HH^*(\text{proj } \mathcal{X})
\]

\[
\text{etc.}
\]

Note: literature compatible with markings: since Kodaira-Spencer

\[
\text{class } \Rightarrow \left[ \begin{array}{c} q \\delta \end{array} \right].
\]

(iii) $H^* = QH^*(X)$

\[
H_* = QH^{m+*}(X) \quad \langle a, b \rangle = \int_X a \wedge b
\]

$K = [D].$

\[\xymatrix{ C_{qs} : \quad QH^*(X) \ar[r] & HH^*(F) \\
& \ar[l] QH^{m+*}(X) \ar[r]_\sim & HH_*(F) \ar[l]}
\]

Prop. $(C_{qs}, oe)$ gives a morphism of closed string data.

Least std. part: $\text{oe}$ respects pairing \[
\langle oe(a), oe(b) \rangle = \langle a, b \rangle \quad \text{Muk}.
\]

Maximal spiritual anatomy $\Rightarrow \Theta^* \subset H^*(\wedge^n \mathcal{F})$

\[
\text{(Kodaira-Spencer class,}
\]

\[
K-S \text{ class non-zero... (this is all that's needed)}
\]

Hence, follow van Hove theory.
Also weak CY str.,
\[ F \xrightarrow{\cong} F^* \xrightarrow{\cong} \mathcal{F} \] + compatibility w/ OC maps.

According to Abo Zaid's smoothness criterion, to say that \( \mathcal{A} \) is smooth & generates \( \mathcal{F} \), suffices to find \( \sigma \in H^H_n(\mathcal{A}) \) s.t. \( OC(\sigma) = 1 \in QH^n \).

In this context, \( OC & CO \) are "dual" maps.

Enough to show that \( CO[D]^n) \neq 0 \) is a generator for \( QH^{2n} \).

That follows, since \( CO \), the map on \( HH \) induced by \( \mathcal{A} \xrightarrow{\cong} \mathcal{B} \), and HKR respect ru, structures & markings.

The fact \( \Theta^n \neq 0 \) implies generation.

\( CO[D]^n \neq 0 \) \( \Rightarrow \) \( CO \) is injective:

\[ a \neq 0 \in QH^*(x) \]

\[ \Rightarrow \exists b \text{ s.t. } a \star b = \omega^n. \]

So \( CO(a) \star CO(b) = CO(\omega^n) \neq 0. \)

Dually, \( OC \) is surjective.

\( F \) smooth \( \Rightarrow \) Mukai pairing non-degenerate \& \( OC \) is an isometry \( \Rightarrow \) \( OC \) is injective (dually, \( CO \) surjective).