1) Two notions of cyclic:

Cyclic subgroup of group $A$

$$C_{\text{cyc}}(A) = (A^{\text{cyc}}, d)$$

$$C_{\text{cyc}}(A) = (A^{\text{cyc}}, \mathbb{Z}^{\text{cyc}}, d)$$

Cyclic category $\mathcal{A}$

$$\mathcal{A} = \mathcal{C} \circ \mathcal{J}, \quad a = \mathcal{C} \circ \mathcal{J}(\mathcal{A}) \rightarrow \mathcal{I}$$

$$(S_{1}, \mu_{1}) \rightarrow (S_{1}, \sqrt{2})$$

**Important:**

$$\triangle \supset \triangle$$ same vertex, but $\text{Hom}_{\mathcal{A}}([m], [n]) = \text{monotone map} \{0 \rightarrow m, \ldots, n \rightarrow 0\}$

$$\text{Aut}_{\mathcal{A}}([n]) = \mathbb{Z}_{n+1}$$
\[ f: (u) \rightarrow [u] \]

Factoring

Ribbons graphs

cyclic ade unreach, \( E_G(v) \).

Given one to surface \( \Sigma \):

\( v \) in \( \Sigma \), then

ribbon cyclic ade.

\( \Sigma \) gives cells in \( \Delta = \Sigma C \)-structure?

well known, e.g., that

\[ B(\text{set of ribbon graphs}) = \bigcup B(\text{top} \text{cyls} \text{ closed gaps}) \]
Crossed simplicial groups (Federovna - Loday)

- Categories related like $\Delta$, which contain $\Delta$ and have similar properties.

$\Delta G \rightrightarrows \triangle$ same objects $[n]$.

If we denote the group

$G_n := \text{Aut}_\Delta(G_n([n]))$, it's required: Can take

any $f : [m] \rightarrow [n]$.

$G_n \ni \phi \circ \Psi \in G_m$.

Examples: (other than $\Delta$ and $\Delta$)

1) Dihedral category

$G_n := D_{n+1} = \mathbb{Z}/2 \ltimes (\mathbb{Z}/n+1)$.

Can be realized in terms of unoriented circles

2) Paracyclic category $\Delta_{\infty}$

$G_n = \mathbb{Z} \ltimes \mathbb{N}$.
3) $N$-cyclic $\nabla N$

$N$-fold cover. Then $G^n = \mathbb{Z}/N(4n+1)$

4) $N$-dihedral $\equiv$

$G^n = D_{N(4n+2)}$

5) Quaternionic $\nabla$

$G^n = Q_{n+1} = \langle n+1, j \rangle \subset H^* = \text{dihedral type subgroup of SU(2)}$

$|Q_{n+1}| = 4(4n+2)$

It's an extension $1 \to \mathbb{Z}/2 \to Q_{n+1} \to D_{4n+1} \to 1$

6) $N$-quaternionic $\nabla N$

$G^n = Q_{N(4n+2)}$

Properties of these:

1) They are self-dual (not including $\Delta$):

$\Delta G^n \rightarrow (\Delta G^n)^\text{op}$

$[n] \rightarrow [n]$.

2) $G^n$ "grows linearly" with $n$, in this sense:

$\text{Hom}(\mathbb{Z}, [n]) \sim \text{Hom}([0], [n]) \xrightarrow{\text{duality}} G^n \times \mathbb{Z} \rightarrow n^2$.

(We are for perception).
3) **Structure groups in 2d**: connective category

\[ H = \text{O}(2) \ \text{s.t.} \ \tilde{p}^{-1}(\text{SO}(2)) \ \text{connected} \]

\[ \text{ker.} \ \text{GL}(2, \mathbb{R}) \]

\[ \text{Conf}_2 = S^1 \times \mathbb{C}^* \]

\[ \text{class. of all these same} \]

\[ G \] - structured \[ C^\infty \] surfaces

\[ \text{moduli space of surfaces} \]

\[ \text{structure} \]

Full list well known:

\[ \mathbb{R} = \text{SO}^0(2) \]

\[ \text{Spin}_N(2) = S^\infty \text{O}(2)_N \]

\[ M = \mathbb{H}, (K^1 \times) \]

\[ \text{moduli space} \]

Each has a moduli spec.
Match exactly, list of graphs.

\[ \Delta \rightarrow \Xi \]

\[ \Delta \rightarrow \Xi \]

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\[ \Delta \rightarrow \Xi \]

\[ \Delta \rightarrow \Xi \]

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\[ \Delta \rightarrow \Xi \]

Claim: For every type of col., \( I \) a sequence of a ribbon graph \( \Gamma \)

some sequence of cyclic ordering \( \Xi \) that classifies

the associated \( M \).

(4) \( \Delta G_n \)-order

\[ \rho : \Delta G_n \rightarrow \text{Set} \]

\[ \begin{array}{c}
\downarrow \\
\{n\} \rightarrow \{0, 1, \ldots, n\}
\end{array} \]

Can speak about "abstract"

find set \( I \), \(|I| = n+1\),

with \( \Delta G_n \)-structure ("order").

Define \( \text{Set}_T \) as set of

tensor action

\[ \text{Set}_T \]

\[ \Delta G_n \rightarrow \text{Set}_T \]

Get a category

\[ \text{Set}_T \]

\[ \Delta G_n \] but more flexible.
Examples:

1) A cyclic order on \( I \) is a ternary relation

\[ \lambda \in I^3 \subseteq \{ (i,j,k) \mid \text{right cyclic order} \} \]

in say what the axioms are, etc.

2) A dihedral order is a 4-ary relation \( \xi \in I^4 \)

\[ \xi \in I^4 \text{ no 3-pt. match but } \]

right "dihedral order", if diagrams meet for

\( K = 9 \) pts.

But for \( |I| = 2 \), 3 more morphisms

\( A \in \mathbb{E} - \text{strake is a } \mathbb{Z}^2 \)-torus,

Sub-example: \( \overline{M}_{0,1} \) (R) real locus.

Times genus 0 w/ 1 marked pts.

Stratified into cells.

Open cells - Stasheff polytopes; labeled by dihedral orders on \( I \). We

have \( 4n^2 \) pts. on \( \mathbb{RP}^1 \).
\[ \text{O}_e \text{-structure graphs } \Gamma. \]

\[ \Gamma(\Gamma) = \text{indegree at } e \quad \text{Ob} = \text{Vert } \Gamma \quad \text{Edges} \]

\[ \text{Morphism } : V \to e \quad \text{inclusion of a } \frac{1}{2} \text{ edge.} \]

A \text{ O}_e \text{-structure is a lift of } F + \text{O}_e. \]

\[ \text{should be with given } \Delta \text{-order.} \]

\[ \text{Examples: } \Lambda \to \text{ Ribbon graphs} \]

\[ \text{dihedral structure:} \]

\[ O(\text{Ed}(e)) \leftarrow O(e) \to O(\text{Ed}(w)) \]

\[ \Gamma \in \text{ surface (oriented or not) has flag structure.} \]
N.B. Naive dihedral graphs (nothing on edges) are not the same.

\[
\begin{align*}
\text{cells in } & \quad \overline{M}(\mathbb{P}^n) \leftrightarrow \text{naive dihedral} \\
\uparrow & \\
\text{trees} & \quad \text{(category of such trees)}
\end{align*}
\]

"Teichmüller spaces of unoriented surfaces" = \{ conform, structures \}

\[
\downarrow
\]

\[
\text{is a cell,}
\]

\[
\text{of [E. Nathan, M. Seppech,]}
\]

\[
\mathcal{B}\left(\text{trivalent graphs}\right) \sim \bigoplus \mathcal{B}(\text{unoriented mapping class groups})
\]

More generally:

1. \[
\mathcal{B}(\text{set-graphs}) \leftrightarrow \bigoplus \mathcal{B}(G^*\text{-mapping class groups})
\]

2. \[
\text{Operad of set-structure}
\]

\[
\text{trees, describes Lie, algebras}
\]

\[
\text{w/ twisted } G^*\text{-structure.}
\]

\[
\text{(see other act by}}
\]

\[
\text{symmetry, w/}
\]

\[
\text{by anti-homomorphic.)}
\]
6. "Invariants" of $G^\otimes$-structured surfaces.

- Numerical.
- Cohomology.
- CY-algebras and categories.
- Categorical $2$-Segal objects.

\[ \Delta \rightarrow \mathcal{C} \rightarrow \text{simple object}, \& \]

\[ \exists f \in \mathcal{C} \] for any polygon.

\[ X_n \xrightarrow{f} X_T = \left( \prod_{\sigma \in T_2} X_2 \right) X_1, \forall \rho \in T_1. \]

**Theorem:** If $\Delta^m \leftrightarrow G$ and $X$ extends to $\Delta^m(G)$ is $2$-Segal, then we can form $X_\mathcal{G} = X_T$, $\forall$ strong $T$, i.e., $T_0$.

G-mapping class group.
Ex: $E = \mathbb{Z}/2N$ - periodic dg - categories

$\sim$ is Morita equivalence.

$G = \text{Spin}_N(\mathbb{Q})$, $A$ a fixed pre-triangulated dg category in $E$.

Then

$X_n = \text{"exact n-hypersimplex"}$ (Waldhausen and M. Gromov).

If $A_{ij}, 0 \leq i < j \leq n, A_{ij} \rightarrow A_{ik}$

has $N$-cyclic str.

$X$ defined for oriented $N$-spin surfaces.

e.g. if $A = C(2N)(\text{Vect}_k)$,

then $X$ top version of Fukaya category of $S$-markings, $(2N$-grade $N$-spin structures).