Ehren, Noncommutative Hitchin System

\[ G = GL_n(\mathbb{C}) \]

\[ X \ \text{sa. poisson} \]

\[ M^S(X) \]

\[ T^* M_{r,d}^S \]

\[ \mathfrak{g} \in H^\bullet(F^\mathfrak{g}_{\infty}) \]

\[ \mathfrak{g} \in H^\bullet(X, \omega_X) \]

\[ \mathfrak{g}(\mathfrak{g}, \phi) = \phi(\mathfrak{g}(\phi)) \]

\[ \text{Sym} \left( \bigoplus_{i=1}^r H^0(X, \omega_X^{\otimes i}) \right) \rightarrow \text{Fun}(T^* M_{r,d}^S) \]

The image is Poisson commutative.

\[ \dim \bigoplus_{i=1}^r H^0(X, \omega_X^{\otimes i}) = r^2(r-1) + 1 = \dim M^S \leq 0. \]

Claim: Hitchin system for \( GL_n \) by \( Y \)-coordinates.

\[ J_e = D^b(X) / < 3 > \]

Assume \( \text{End}_X(\mathfrak{g}) = \mathbb{C} \)

Easy to understand this quotient category: fix \( p \in X \).

1. Then, \( 0_p \) is a split generator of \( J_e \).

2. \( \text{REnd}_{J_e} (0_p) \) is just Grassmannian algebra (only 0 cohomology).

Call it \( B_p \).
Eulerian quotient to see
\[ \operatorname{Ext}^3_\mathcal{O} \mathcal{E} \otimes \mathcal{E}(\mathcal{O}) \to \operatorname{Ext}^2_\mathcal{O}(\mathcal{O}, \mathcal{O}) \]
which will \[ \operatorname{Ext}^1_\mathcal{O}(\mathcal{O}) \]

Propertes:
I) Each \( \mathcal{E} \) is constructively finitely presented (by Töen)
   (e.g., finitely presented and homologically smooth)
   (not evident from definition).
II) \( \mathcal{E} \) is actually \( \text{Quillen smooth} \),

(Def: A assoc. algebra, A Quillen smooth if
\[ \Sigma A \text{ } - \text{projective over } A \otimes A^\text{op} \]
  \text{bundle of differentials} \[ \Sigma A \to A \otimes A \to A \to 0 \]
  \text{lifting property for } \Sigma \mathcal{E} \text{ on } \mathcal{E} \text{ extensions:
  }\quad A \to B \]
\[ \mathcal{E} \to \mathcal{B} \text{ } \Sigma \mathcal{E} \text{ extension} \]

Pf.
\[ D^+(X) = \operatorname{Perf}(\mathcal{A}_X) \]
\[ \mathcal{A}_X - \mathcal{R}(\mathcal{E} \otimes_\mathcal{O} \mathcal{O}) \]
Then have morphism \( \mathcal{A}_X \to \mathcal{B}_\mathcal{E} \).
To show Quillen smooth, need \( \operatorname{Ext}^2_\mathcal{B}_\mathcal{E}(\mathcal{B}_\mathcal{E}, \mathcal{B}_\mathcal{E} \otimes \mathcal{B}_\mathcal{E}^\text{op}) = 0 \) and show on \( \Delta x \mathcal{A}_X \) shifted by \( x \)
\[ \operatorname{Ext}^2_\mathcal{A}_X(\mathcal{A}_X, \mathcal{B}_\mathcal{E} \otimes \mathcal{B}_\mathcal{E}^\text{op}) \]
which is 0 by duality results on \( \mathcal{E} \).
General story:

A Quillen-smooth $\Rightarrow$ $\text{Rep}_V(A)$ are smooth affine schemes.

$V$ f.d. vector space.

$\Downarrow$

$\text{Rep}_V(A)$ an affine scheme w/

$\text{Rep}_V(A)(R) = \text{Hom}_{\text{alg}}(A, R \otimes \text{End}(V))$.

Get a natural $GL(V)$ action on $\text{Rep}_V(A)$.

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Interested in finite dimensional modules over $B = \text{mod}_A - B^\Sigma$

Prop: If $\Sigma$ is unstable, then $\Rightarrow$ mod $\text{f.d.} - B^\Sigma = \emptyset$.

why? ps-perfect modules $(B^\Sigma) = \Sigma^\perp \cdot \text{Coh}(X)$

If $\Sigma$ unstable, then $\Rightarrow \Sigma^\perp = \emptyset$

(Pf: If $\text{Ext}(\Sigma, F) = 0$ $\Rightarrow$ slope $F = \text{slope}(\Sigma) + g - 1$

But if unstable, have any $\text{slope}(\Sigma_1) > g - 1$

have $SS(c) \subset \text{Coh}(X)$

category of semi-stable vector bundles with slope eq.

$SS(c)$ - abelian of finite length, & simple objects are pure plus stable bundles.
It's clear that \( L \).

(b) \( \text{mod } \mathcal{D}, \mathcal{B} \subseteq S S (A + g - A) \).

I see subcategory. (Q: how do you see finite \{free \text{ pure objects \& pure objects?} \}

\( \text{Ans. using internal \text{-}structures } \).)

Standard calculus for smooth objects:

\( A \) = Quillen-smooth algebra.

\( S^n_A := (S^n_A)^{\bigotimes n} \rightarrow \) \( n \)-th tensor power over \( A \) also a superalgebra.

DeRham differential:

\[ d : S^n_A \rightarrow S^{n+1}_A \]

First:

\[ d : A \rightarrow S^0_A. \]

\[ d(a) = a \otimes 1 - 1 \otimes a. \]

Then each \( S^n_A \) can be given as \( a d a, \ldots, d^n a. \)

Not interesting: see sheafology in day 0.

\( w, d(\partial a) = 0. \)

Next:

\[ DR^n(A) = S^n_A / \left[ S^n_A, S^n_A \right] \]

Get \( d : DR^n A \rightarrow DR^{n+1} A. \)

with representative spaces, get a map of complexes:

\[ (DR^n A, d) \rightarrow (S^n_{\text{Rep}_V(A)} , d). \]
Given by usual map

\[ A \rightarrow \text{Der}(\text{Rep}_V(A)) \otimes \text{End}(V), \text{ have the face.} \]

\[ g \mapsto H^0(A) \rightarrow C[\text{Rep}_V(A)] \]

\[ \Pi \]

\[ A/\{A, A\}. \]

Rule: \[ \text{Der}(A, M) = \text{Hom}_{\text{Mod}(\mathcal{R}_A)}(\mathcal{R}_A, M) \]

\text{Der}(A, A)

\text{Standard calculus:}

\text{Give} \, \Theta \in \text{Der}_p A, \text{ substitute gives}

\[ i_{\Theta} : DR^k A \rightarrow DR^{k-1} A, \text{ \& Lie deriv. relations}. \]

A symplectic structure on A is a closed 2-form

\[ \omega \in DR^2(A), \omega = 0, \text{ giving an isomorphism} \]

\[ \text{Der}_A \rightarrow \otimes DR^2 A. \]

If \((A, \omega)\) symplectic, then CA needs to be compactly.

\((\text{Rep}_V(A), \text{Tr} \cdot \omega)\) -- symplectic affine variety.

\[ (a, x) \rightarrow (a \cdot x) \rightarrow a, \otimes \text{Der}_A. \]

\[ a \in A \rightarrow a \in DR^1 A \rightarrow a \cdot \Theta_a \text{ Der}_A. \]

\[ (a, b) \rightarrow \Theta_a(b). \]

\[ A/\{A, A\} \rightarrow C[\text{Rep}_V(A)] \text{ isomorphism of Lie algebras.} \]
\[ \text{Der}_A = \text{Der}(A, A \otimes A) = \text{Hom}_{A \otimes A^{op}}(R_A, A \otimes A) \]

By definition,
\[ T_A^* := T_A(\text{Der}_A) \]

Hence, algebra over \( A \) of this bimodule.

\[ \begin{array}{c}
0 \rightarrow T_A^* A \otimes A \stackrel{\delta_A}{\rightarrow} S_{T_A^* A} \\
\end{array} \]

If implies that if one has:
\[ \text{Der}_A \otimes A \rightarrow \Omega^1 \]

\[ \text{Hom}_{A \otimes A^{op}}(R_A, R_A) \xi \rightarrow T_A^* A \otimes A \rightarrow \Omega T_A \]

and \((T_A^* A, \omega_A)\) is symplectic manifold variety.

\[ E: A \rightarrow E_\xi(A) \]

\[ \text{If } \text{Rep}_\nu(A) = \text{Der}(A; E_\xi(A)) \]

\[ = \text{Hom}_{A \otimes A^{op}}(S_{E_\xi(A)}, E_\xi(A)) \]

\[ \text{If } \text{Rep}_\nu(A) = \text{Hom}_{A \otimes A^{op}}(D_{E_\xi(A)} E_\xi(A)) \]

\[ \text{Rep}_\nu(T_A^*) = T^* \text{Rep}_\nu(A) \]