$X$ smooth alg. variety, $n$ algebraic volume form.

Hochschild cohomology $H^*(X) = H^*(X, A^*TX)$. This comes a product (and a Lie bracket), $c \in H^0\left(\mathcal{O}_X\right)$ unit.

It also comes the BV operator $\Delta$, defined as:

$$
\Delta: \bigwedge^n TX \xrightarrow{\gamma} \bigwedge^{n-1} TX \xrightarrow{\gamma} \bigwedge^{n-2} TX
$$

We also have cyclic cohomology ($= S^1$-equivariant Hochschild cohomology).

Take $u = \text{fin.-var. of degree 2}$. Then,

$$
H_{S^1}^n(X) := H^n(X, \mathcal{O}_X) \xrightarrow{u^*} \bigwedge^2 TX \xrightarrow{u^*} TX \mathcal{O}_X \mathcal{O}_X
$$

Obviously,

$$
\sim \xrightarrow{u^*} H_{S^1}^{n-2}(X) \xrightarrow{u^*} H_{S^1}^{n-2}(X) \xrightarrow{u^*} H_{S^1}^{n-2}(X) \xrightarrow{u^*} H_{S^1}^{n-2}(X)
$$

Moreover,

$$
H_{S^1}^{n+2}(X) \otimes \mathbb{C}[u] \cong H_{S^1}^{n+2}(X) \otimes \mathbb{C}[u, u^{-1}] \cong H_{S^1}^{n+2}(X) \otimes \mathbb{C}[u, u^{-1}]
$$

Now, assume that $X$ is in fact a family of curves depending on a parameter $\varepsilon$.

Then, there is a canonical class

$$
k \in H^2(X, TX) \cong HH^2(X) \quad \text{(Kodaira-Spencer class)}
$$

measuring the variation in $\varepsilon$-direction.

Gauss-Manin "connection":

$$
\Gamma: HH_{S^1}^n(X) \xrightarrow{\Gamma} HH_{S^1}^{n+2}(X);
$$

$satisfies:

$$
\begin{align*}
\Gamma(f(t) \cdot x) &= f(t) \Gamma(x) + u \left(\partial_t f\right) x, \\
\Gamma & \text{ fits into}
\end{align*}
$$

\[
\begin{array}{ccc}
H_{S^1}^n(X) & \xrightarrow{\Gamma} & H_{S^1}^{n+2}(X) \\
\downarrow u = 0 & & \downarrow u = 0 \\
HH^*(X) & \xrightarrow{k_*} & HH^{*+2}(X)
\end{array}
\]
Remark: Both $e$ and $k$ lift to classes in $HH^k_{SA}(X)$ (obvious for $e$, and use $\Gamma'(e)$ for $k$).

Assumption: $k=0$.

If the assumption holds, there are connecting maps

$$
\begin{align*}
HH^k_{SA}(X) \xrightarrow{\nabla^S_k} & HH^k_{SA}(X) \\
\downarrow \quad u = 0 & \downarrow \quad u = 0
\end{align*}
$$

such that $u \nabla^S_k = \Gamma$

"can divide by $u$ now."

(Advantage: $u$ is a row, so we can write down diffs which are non-linear.)

Consider $X$ independent of $t$, but with a $t$-dependent $\eta$. Then,

$$
\begin{align*}
\Lambda^i \mathcal{R} \to \Omega^{-i} \xrightarrow{\partial_t} & \Omega^{-i} \xrightarrow{\Xi} \Lambda^i \\
& \downarrow \nabla
\end{align*}
$$

In particular,

$$
\nabla(e) = \frac{\partial_t \eta}{\eta} \in H^0(X, \mathcal{O}_X) \subseteq HH^0(X).
$$

(call this $a$);

and

$$
\nabla(\xi) = \partial_t \xi + a \cdot \xi.
$$

Specifically, look at

$$
\pi: \tilde{X} \longrightarrow \mathbb{P}^A = \mathbb{C}^{u \times v + 5} \quad \text{a "Calabi-Yau fibration,”}
$$

$$
k = \pi^* \mathcal{O}_{\mathbb{P}^A}(1). \quad \text{(Simpler thing: until elliptic surface)}.\n$$

($\Rightarrow$ all fibers CY, $\delta$ complement of $\pi^{-1}(p)$ is CY too!) .

In particular, there is an algebraic volume form $\rho_0$ with a pole at $\pi^{-1}(\infty)$.\n

Take
\[ X = \bar{X} \setminus \pi^{-1}(\frac{1}{t}) \, . \]

This comes as an algebraic volume form \( \omega = \omega_0 \frac{1}{1-tx} \). (work of g on plane behind of oo)

The assumption is always satisfied! (work of g always ahead of oo)

Formally no difference between \( \mu \), (having a pole at \( p \) & infinitely near \( p' \).)

We have
\[ q \overset{\text{def}}{=} \frac{\alpha \cdot l}{l} = \frac{\pi}{a - t\pi} \, . \]

Note \( \alpha \cdot \beta = (1 - t\pi)^2 = q^2 \). Hence,
\[ \nabla (\nabla e) = 2(\nabla e)^2 \]

(Flexibility in parametrization for \( q \) & \( t \); introduces lower order terms. This characterizes normalizations of \( q \) and \( t \).)

E open symplectic manifold, \( \alpha_2(E) = 0 \).

\[ \mathfrak{SH}^*(E) \text{ symplectic cohomology; } \mathbb{Z} \text{-graded vector space over } \mathbb{R} \]
\[ \mathfrak{K} = \left\{ f(q) = \alpha_0 q^m + \alpha_1 q^{m+1} + \cdots \right\} \quad a_i, m_i \in \mathbb{R} \, . \]

(Not too picky in \( \mathfrak{K} \); want to take \( \mathfrak{K}_2 \) !)

This has the structure of a ring, Lie algebra, and comes a BV operator. It comes with a map
\[ H^*(E; \mathfrak{K}) \rightarrow \mathfrak{SH}^*(E) \quad \text{(''acceleration'')} \]
\[ 1 \rightarrow e \quad \text{unit class} \]
\[ q^{-1} [w_E] \rightarrow k \quad \text{''Kodaira- Spencer class''} \]

There is an equivariant version \( \mathfrak{SH}_2^*(E) \), w/ \( \rightarrow \mathfrak{SH}_2^*(E) \rightarrow \mathfrak{SH}_2^*(E) \rightarrow \mathfrak{SH}^*(E) \rightarrow \quad \text{(''equivariant acceleration'')} \)

and there is a map \[ H^*(E; \mathfrak{K})[u] \rightarrow \mathfrak{SH}_2^*(E) \]
Theorem (Albers - Cieliebak - Frenkel, Zhao):

\[ H^*(E) \cong SH^*_S(E) \]

\[ \text{becomes an isomorphism after inverting } \mathbb{Q} \]

(Proven easier in this case, b/c isomorphism with \[ SH^*_S(E) \\] as usual provided \[ SH^*(E) \text{ is torsion-free.} \])

Also,

\[ SH^*_S(E) = \left\{ \begin{array}{ll} 0 & * < 0 \\ SH^0(E) & * = 0 \end{array} \right. \]

Any class in \[ S^0 \] has a unique equivariant lift.
From the compactification $\overline{E}$; we get

$$s \in SH^0(E) \quad \text{("Barman-Shenon class")}.$$

(Refl: should be defined as an equivalence class, but doesn't matter by prev. remark.)

(Refl: after throwing away a torsion, which $H^0(E)$ class should this correspond to?)

**Lemma:** \[ u_{\text{seq}} = z^{(s)} \mid_{E}, \text{ where} \]

$$z^{(s)} \in H^2(\overline{E}; \mathbb{K}) \text{ counts holomorphic sections.}$$

(Invoking $u$, \( s_{\text{seq}} = u^{-1} z^{(s)} \mid_{E} \); but the -o invoking!)

As a consequence, \( z^{(s)} \mid_{E} \) lies in the kernel of the acceleration map
(two $SH^*(E)$).

(As a conjecture, this was in the literature already.)

**Conjecture:** \[ 2u^2(s \circ s)_{\text{seq}} = (z^{(s)} \mid_{E}) \ast (z^{(s)} \mid_{E}) \]

\[ \uparrow \quad \text{product in} \] \( SH^*(E) \)

\[ 
\begin{align*}
&\begin{bmatrix}
\end{align*}
\]

\[ 
\begin{align*}
14 z^{(\sigma, \lambda)} \mid_{E} + u^2 z^{(s)} \mid_{E} \quad \uparrow \\
&\text{counts bidegrees (rat. class of degree 0): fn. of } \sigma. \\
&\text{(deg. 2 over the base).}
\end{align*}
\]

(Refl: These are the first two instances of a series of formulas; no idea what the higher ones are!)

**Assumption:** There are $\psi, \eta \in K$ such that

$$g^{\lambda} [\omega_{\overline{E}}] = \psi \cdot z^{(s)} - \eta [M] \in H^2(\overline{E}; \mathbb{K}).$$

This implies \( g^{\lambda} [\omega_{\overline{E}}] = \psi \cdot z^{(s)} \mid_{E} \) lies in the kernel of acceleration.

Therefore, \( k = 0. \)

**Theorem (Assuming conjecture):** If the assumption holds

(on next page)
Theorem (assuming competitive): If the assumption holds, the connection $\nabla$ on $S(E)$ satisfies:

$$\nabla \nabla (e) - 2 (\nabla e) \cdot (\nabla e) + \left( \frac{\partial^2 \psi}{\partial^2} - \eta \right) \psi = 0.$$

Recall, we had the 0th $f''(t) = 2f'(t)^2$; but lower order terms here. As part of general theory, can reverse $\psi$ to get back to $\delta$; goes the origin map.