SYZ - Intro to Log Geometry

log space = space + log str. (scheme)

Motivation: 2 classical problems

1) Compactification

\[ \mathcal{U} \text{ open (not open) smooth} \]

\[ \mathcal{U} \hookrightarrow X \text{ compactification} \]

\[ \mathbb{A}^1 \hookrightarrow \mathbb{P}^1 \]

Original idea: study a pair \( (X, \mathcal{U}) \) \( \Rightarrow \) "specifying log str." on \( X \).

Then (Gorthendieck):

\[ H^i(U^\circ, \mathcal{O}_U) = H^i(U, \Omega^\circ_U) \]

(Pf: Find nice compactification, apply Hodge theory)

\[ \varpi_U = 0, \quad \Omega_U = 1\text{-forms} \]

Interesting cases:

- \( \mathcal{U} \) affine

\[ U \subseteq \mathbb{C}^n \quad \Rightarrow \quad H^i(U, \mathcal{O}_U) = 0, \quad i > \dim \mathcal{U} \]

(even though \( \dim \mathcal{U} = 2 \cdot \dim \mathcal{U} \)).

Intermediate objects: "log De Rham complex on \( X \)" and this comes a Hodge str.

\[ H^i(U, \Omega^\circ_U) = H^i(X, \Omega^\circ_U) \]

(2) Degeneration
Family
\[ f: X \to S \quad \text{(e.g. } S = \text{curve} \) \\ \[ S = \text{neighborhood of } \Theta \text{ in } C) \]

Assume: \( f \) is proper, \( X \) is irreducible. (If \( S \) curve, \( \Rightarrow \) flatness)

\( f \) is smooth away from \( \Theta \in S \), but \( X_0 \) is singular.

\[ X = \sum (x^4 + y^4 + z^4 + w^4) - 4xyzw = 0 \] 
\[ \cap \] 
\[ \mathbb{P}^3 \times S \]

\( X_{S \neq 0} \) is a \( K3 \) surface

generic

\[ X_0 = \text{Unord. planes in } \mathbb{P}^3 \]

Try to resolve \( X_{S \neq 0} \) from \( X_0 \) w/ extra data...

(3) = (1)+(2) COMPACTIFYING MODULI SPACES.

Log Structure

A log structure on \( X \) is a sheaf of commutative monoids \( M \) & a map \( \alpha: M \to \mathcal{O}_X \) (of monoids)

such that:

\[ \alpha: \mathcal{O}_X \to \mathcal{O}_X \] 

(\( \Rightarrow \) have \( \mathcal{O}_X^* \to M \to \mathcal{O}_X \)), so \( M \) is some sort of enrichment...
Examples:

1. **Compactifying Log Structure**
   \[ X \supset U \text{ open} \]
   \[ M = \{ f \in \mathcal{O}_X \mid f|_U \in \mathcal{O}_U^* \} \]
   The map \( \omega \) is just restriction.
   \[ \mathcal{O}_X^* \subseteq M \subseteq \mathcal{O}_X \]

2. **Degenerating Log Str.**
   \[ f : \mathfrak{X} \rightarrow S \]
   as before.
   \[ x_0 \in \mathfrak{X} \supseteq x_0^0 = f^{-1}(S^0) \]
   \[ \downarrow \quad \downarrow \]
   \[ 0 \in S \supseteq S^0 = S \setminus 0 \]
   "016" \( \Delta^0 \)

   Look at \( M_{x_0}/\mathfrak{X} \)
   log str. on \( \mathfrak{X} \)

   Pull back this log structure along \( i : x_0 \rightarrow \mathfrak{X} \) to
   get a log str. on \( x_0 \).

3. **Pull Back**
   \[ f : Y \rightarrow X, \quad M \rightarrow \mathcal{O}_X \text{ (log. str.) on } X. \]
Define \( N':=f^{-1}(\mathcal{U}) \xrightarrow{f^{-1}(\mathcal{O}_X)} f^{-1}(\mathcal{O}_Y) \xrightarrow{\beta'} \mathcal{O}_Y \).

Problem: \( N' \) doesn't contain \( \mathcal{O}_Y \).

\((\beta')^{-1}(\mathcal{O}_Y^*) \xrightarrow{\text{(universal constructor exists for \( \mathcal{O}_Y^* \))}} \mathcal{O}_Y^* \xrightarrow{\text{fles}} \mathcal{O}_Y \)

(log str. by general nonsense.)

Sub-example:

\[ x = A^2_{xy} \xrightarrow{(x,y) = xy} A^*_t \xrightarrow{f} S \]

\[ U = A^2 \xrightarrow{\text{log str. on } A^2} A^*_t \]

Restrict to \( \mathbb{A}^2 \):
The map

\[ N \rightarrow (0 \rightarrow ) \rightarrow x, y \rightarrow y, \]
\[ (x, y) \rightarrow y, (x, y) \text{ is not injective!} \]

(N.B. We've remember that \( x \cdot y \) is not zero on \( \mathbb{R} \)).

How to "see" \( M \)?

\[ 1 \rightarrow \mathcal{O}_x \rightarrow M \rightarrow \overline{M} \rightarrow 1. \]

"ghost sheaf."

Sheaves \( \overline{M} \) are usually finitely generated monoids.

E.g.,

\[ \begin{align*}
\mathcal{O}_x & \rightarrow \mathcal{M} \rightarrow \mathcal{N} \rightarrow 0 \rightarrow \\
\text{order of vanishing?} & \quad \text{order of vanishing?}
\end{align*} \]

\[ \mathcal{N} < y > \]

(4) affine toric varieties \( \sigma = \text{cone } \mathcal{M} \),

\[ \sigma \cap \text{lattice } \phi = \text{cone } \text{lattice } \phi \cap \text{cone } \]

\[ P = M \cap \sigma \text{ is a finitely generated monoid.} \]

\[ \mathcal{O}(\mathcal{C}[P]) = \mathcal{O}(x, y, z)/\langle xy - z^2 \rangle. \]
Ex. 6) Cusp curve
\[ X = \{ x^2 = y^3 \} \]

\[ M = \{ f \in O_X \mid f(\gamma \circ \Theta) \neq 0 \} \]

(5) \[ \frac{\mathcal{A}^2 \setminus X}{\mathcal{A}^2} \]

Natural log structure on affine variety \( X = \text{Spec } C[P] \).

Log str.: \( M = \) subsheaf of \( O_X \) generated by \( O_X^* \) & \( P \).

Equivalently:
\[ M_{\mathcal{U}} \times X \quad \text{U = open torus orbit} \]
\[ U = \text{Spec } C[M] \]

Motto: Good log structures are those built from these by pull back.

Log differentials

\[ X \xrightarrow{\Delta} X \times X \]
\[ D = x^1 u \]

\( u \) log vector fields

\( x \) vector fields on \( X \) tangent to \( D \).

Def: A log derivation into \( \mathcal{E} \) is

\[ (\text{log derivation into } E) \quad X \xrightarrow{\partial} E \]
Differentials allow "log poles" along $D$.

If $D$ loc. given by $X = 0$, should define

$$ \frac{dx}{x} = d(\log x) $$

Given $(X, M)$, define $\mathcal{Z}^i_{(X, M)}$ log differential forms

$$ \mathcal{Z}^i_{(X, M)} \text{ log differential forms} $$

Can compute $HI^{i}(X, \mathcal{Z}^i_{(X, M)})$ should be "right" cohomology.

(e.g. cohomology groups of general fiber)

(b can also see unramified action).

Vec. fields: denote

$$ (X, M) \mapsto X_{\log} \rightarrow X $$

Kato-Nakayama space

$$(X, M) \mapsto X_{\log} \rightarrow X $$

Skipper...
GROSS - SIEBERT Program

Toric degeneration

\[ f : \mathcal{X} \to \mathcal{S} \]

Take critical fibre \( \mathcal{S} \neq 0 \), \( \mathcal{X}_0 = \text{min of toric variety} \)

(e.g. \( s(\mathcal{X}^Y, t + W_Y) = \mathcal{X}^Y + W \))

Get a "toric log (\mathcal{X}, Y)" structure

\[ \mathcal{X} = \text{min of toric variety} \]

\[ M = \text{log str.} \]

\[ \mathcal{X}^+ = (\mathcal{X}, M) \to (p^+, \mathcal{N} \to \mathcal{C}) = p_+^+ \]

\( \mathcal{N} \to \mathcal{C} \)

Polyhedra \( \leftrightarrow \) vertices of \( \mathcal{X} \)

Fan at vertex \( \leftrightarrow \) irreducible component

Logarithmic degeneration data

\((\mathcal{B}, \mathcal{P})\) polyhedral decompos.

\( \mathcal{B} = \text{manifold glued from lattice polyhedra} \)

Ex. of \((\mathcal{X})\)

\[ \mathcal{X} \]

\[ \mathcal{M} \]

\[ \mathcal{N} \]

\[ \mathcal{R} \]

Take \( \mathcal{P}_V \) = preimage of \( \mathcal{I} \)

From last time.
polarized log CY \rightarrow \text{ample} / X \rightarrow \text{polarized degeneration.}

\text{Can go back wards, for natural involution.}

\text{Discrete Log Transform.}

\text{Character interchanger log st.}

\text{data & polarization. data (')}