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Fix $K$ a field of char. 0.

Let $B$ be an integral matrix $(n \geq n)$; w/ initial $n \times n$ block skew-symmetric.

$$
\begin{pmatrix}
  n \times n & \| \\
  & n \times (m-n)
\end{pmatrix}
$$

Associate variables $A_{ij}$, $B_{ij}$ to the columns of $B$.

$(B, \{A_1, \ldots, A_m\})$ is called a seed datum. New seeds are obtained by applying mutations.

Let $1 \leq k \leq n$. The $k^{th}$ mutation is

$$B_{ij} = \begin{cases} 
- B_{ij} & \text{if } k \in \{i,j\} \\
B_{ij} & \text{if } B_{ik} B_{kj} \leq 0 \text{ and } k \notin \{i,j\} \\
B_{ij} + |B_{ik}| B_{kj} & \text{if } B_{ik} B_{kj} > 0 \text{ and } k \notin \{i,j\}
\end{cases}$$

Also, $A_i' = A_i$ if $i \neq k$.

and $A_k A_k' = \sum_{j : B_{kj} > 0} A_j B_{kj} + \sum_{j : B_{kj} < 0} A_j B_{kj}$. (the exchange relation).

A priori, $A_1', \ldots, A_m' \in K(A_1, \ldots, A_m)$

$$\in K[A_1^{\pm 1}, \ldots, A_m^{\pm 1}]$$

by this formula.

This gives new seed datum $(B', \{A_1', \ldots, A_m'\})$.

Can repeat this many times...

Def: The (upper) cluster algebra associated to the initial seed datum is the $K$-subalgebra of $K(A_1, \ldots, A_m)$ of universal Laurent polynomials, i.e.

clusters of which are Laurent polynomials in any mutated seed.

Thm: (Fomin-Zelevinski): The Laurent phenomenon: $A_1, \ldots, A_m$ are universal Laurent polynomials.

Variable $A_1, \ldots, A_m$ for any mutations.
Fun case: \[
\left( \begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array} \right)
\] 5 distinct seeds; sort of a miracle that it all works.

We'll introduce techniques from mirror symmetry to prove a number of conjectures in cluster algebras:

- Positivity of Laurent phenomenon: Expressing $A_i$ in terms of initial seed vectors always gives a Laurent poly. with positive coefficients.
  (Lee-Schiffler; skew-symmetric case).
- Sign coherence: \[
B = \begin{pmatrix}
\text{skew-symmetric} & \\
& I \\
\end{pmatrix}
\]
  rows of this $n \times n$ block identify as $k$ gp. of modules over a quiver.
- Construction of canonical bases
  (original motivation: looking at cluster-ideal canonical bases).

Scattering diagrams: Fix $M = \mathbb{Z}^m$, $R_M = M \oplus \mathbb{R}^n$.
Fix $B$, and assume that the row vectors $V_i \in M$.

\[
B = \begin{pmatrix}
V_1 \\
\vdots \\
V_m
\end{pmatrix}
\]
are a strictly convex cone $\sigma \subseteq R_M$.
(e.g. principal root case).

$P = M \sigma$ (finitely generated) polytope.

\[
\hat{K}[P]
\]
completion of $K[P]$ along unique maximal maximal ideal $P \not\mid \mathbf{0}^3$.

Def: A wall is a pair $(d, f_d)$ where $d$ nonstrictly convex.
- $d$ is a codimension rational polyhedron core in $R_M$ with
  $e$ an element $\mathbf{0} \in P \not\mid \mathbf{0}^3$ s.t. $d = d - R \geq 0$.
e.g. \[ \begin{array}{c}
\text{one fluid by ans.}
\end{array} \]

we say \( d \) is \textit{meaningful} if \( d = d \oplus R_{no} \).

e.g. \[ \begin{array}{c}
\text{Def: A \textit{scattering diagram} } D
\text{ is a collection set of walls } \mathcal{W} \text{ such that }
\forall k > 0, \# \{ (d, f_d) \in D \mid f_d \equiv 1 \mod m_k \} < \infty.
\end{array} \]

If \( D \) is a \textit{scattering diagram},
\[ \text{supp } D = \bigcup_{(d, f_d)} \text{Sing } D = \text{locus where } \text{supp}(D) \text{ is not a manifold}. \]

Let \( \theta \) be a path in \( M_{\mathbb{R}} \setminus \text{Sing}(D) \)
with endpoints in \( M_{\mathbb{R}} \setminus \text{supp}(D) \).

Then, the \textit{path-ordered product}
\[ \Theta_{\mathbb{R}, D} = \text{Aut}_{\mathbb{R}}(K_{\mathbb{P}}). \]

Crossing one wall \( (d, f_d) \) at time \( t_0 \), we get \( \Theta_{\mathbb{R}, d} \) defined by
\[ Z^m \rightarrow Z^m \cdot f_d^{<n_0, m>} \text{ where } n_0 \in N \text{ annihilates } d. \]

We then take composition in the order of walls crossed.

**Key example:** Let \( e_1, \ldots, e_m \) be the standard basis for \( N \), the matrix
\[ D_{in} := \{ (e_i, 1 + z \bar{v}_i) \mid 1 \leq i \leq n \}. \]

Thus: (G-Siebert '07, Katzarkh-Siebertman) \( K_S '04 \) in 2-planes
such that \( \overline{D} \setminus D_{in} \) has no meaning walls and \( \Theta_{\mathbb{R}, D} = id \) for loops \( \gamma \) for which this is defined.
E.g.,

\[ B = \begin{pmatrix} 0 & l \\ -l & 0 \end{pmatrix}, \quad l \in \mathbb{Z}_{>0}. \]

\[ \begin{array}{c}
\text{l=1} \\
\text{O} \\
\text{l=2} \\
\end{array} \]

\[ \begin{array}{ccc}
1/2 & -1/3 & -1/5 \\
\end{array} \]

\[ \text{l>3, will behave.} \]

\[ B = \begin{pmatrix} 0 & -2 & -2 \\ -2 & 0 & -2 \\ -2 & -2 & 2 \end{pmatrix} \]

Can show: 1-1 corres. between related seeds and chambers, if one can reach first ordinate.

Key point: All f's have positive integer coefficients.

Also: any to construct from Settop diagrams, for every integral point, union, if each chambers related by well-order, orthogonally.
Get all of the data
state $\mathcal{A}_1 \to \mathcal{A}_2$, give description which is non-linearly positive.

Use optimal use of symplectic homology.

Ref: picture of Frk & Goodman.