Day 4, Talk 1: Tim Perutz

Fixed point Floer homology (of complex Dehn twist)

\((M^2, \omega), \eta \in \text{Aut}(M), \eta = 1 \text{ near } 2M\)

\[ HF_\eta(m) = H(\text{CF}_\eta(m \circ \phi)) \text{ Hamiltonian} \]

\[ CF_m(q \circ \phi) = H\text{Fix}(q \circ \phi) \text{ where } H = \mathbb{Z}/2 \text{ -- take } \begin{cases} M \text{ exact, } \eta \text{ exact or } M, \eta \text{ monotone} \\ \text{perturb } \eta \text{ so small } \rightarrow \text{ perturbations, or large Reeb flow perturbations } \end{cases} \]

\[ \Rightarrow HF_\eta(i\theta) = H \Omega M, \text{ otherwise } H_\eta(M, 2\pi) \]

Take a seq. of parameterized Lag. spheres \((V_v, -V_v)\) in \(M\).

These should be either exact (if \(M\) is) or balanced (monotone arcs)

Balanced — choose pair \((\Theta, A)\), \(\Theta \in \Omega^1(M), A \in \Omega^1(S(\kappa_M))\)

\[ \chi F_A = \omega - d\Theta \]

A Lag. \(L \subset M\) is balanced w.r.t. \((\Theta, A)\) if

\[ [\chi \sigma A + \Theta L] = 0 \in H^2(L; \mathbb{R}) \text{ (indep. of } \sigma) \]

Key fact: Balanced deformations of a Lag. submfd are Hamiltonian

(Analogous to exact Lag.):

\[ S(\kappa_M) \mid L \]

\[ L \]

\[ \sigma \]

\[ \Downarrow \]

\[ p \]
Can form $\mathcal{A}$ so categories $\mathcal{A}$ - directed
$\mathcal{B}$ - fill
with objects $V_1, \ldots, V_N$ over $K = \mathbb{Z}/2$.

Rule: When $M$ is a surface with area form, one can check in "algorithmic" fashion whether a given finite set of curves is "balanced" for some $G/A$.

(Typical system of eqns.) "Gauss-Bonnet" picture

For our case, Euler characteristic of both should be 0.

Take $\eta = \tau V_1 \circ \cdots \circ \tau V_N$ of Dehn twist along the $V_i$'s.
So $\eta$ is monodromy of fibered fibration

\[ E \xrightarrow{\pi} M \]

Goal: Describe $HF_*(\eta)$.

(Gen) (Seidel)

Formulation 1 (2004). A natural LES

\[ H_* E \rightarrow HF_*(\eta) \]

\[ HH_*(A,A) \]


\[ HF_*(\text{id}_M) \rightarrow HF_*(\eta) \]

\[ HH_*(A,B) \]

maybe up to shift.
Work in progress should prove both these conjectures.

What's the difference?

\[ CC_\circ (A, B) = R \oplus \bigoplus_{i \in \mathbb{N}} \operatorname{C} \circ \circ \operatorname{C}(V_i, V_j) \oplus \omega \operatorname{C} \circ \circ \operatorname{C}(V_{i_0}, V_{j_0}) \oplus \omega \operatorname{C} \circ \circ \operatorname{C}(V_{i_1}, V_{j_1}) \]

\[ \text{Difference is } K \oplus K[-n] \]

Difference is \( K[-n] \), so \( H_{-n}(A, B) \) is another \( K^{[n]} \) but \( H_{-n} \in \operatorname{Geom} \)

So it boils down to a choice of where you put the \( K^{[n]} \), either in \( H_0(A, B) \) or in \( H_{-n} \).

### Appendix to Formulating Formulation 2

**Algebraic**

\[ \text{Gr} (M) = (A_M \times T) \oplus \omega ((A_M \times T) \Delta A_M) \]

\[ \begin{array}{c}
L \in \mathbb{A} \wedge L \in \mathbb{A} \\
\text{Fibered Delm twist along} \\
M \times V_i, \text{ (spherically fibred cosubripic)}
\end{array} \]

Prove that if \( CC \circ \circ \operatorname{C} \circ \circ \operatorname{C} \) is spherically fibred cosubripic, then

\[ \operatorname{C} \circ \circ \operatorname{C} \circ \circ \operatorname{C} : \text{cone} \left[ \text{cone} \left[ \operatorname{C} \circ \circ \operatorname{C}(M, T, L) \right] \rightarrow \text{cone} \left[ \operatorname{C} \circ \circ \operatorname{C}(M, T, L) \right] \right] \]

(Act on \( M \)).
Actual approach:

Define homomorphisms:

\[ \phi : G \times A \rightarrow G' \]
and \[ \psi : G' \rightarrow G' \] by

\[ \phi(x, a) = \psi(x) \cdot a \]

- Get at fixed points per cycle of rotation or reflection
- Check if permutations have the same cycle type

Look for fixed points with steps:

\[ T_0 \rightarrow T_1 \rightarrow T_2 \rightarrow \ldots \]

\[ G \rightarrow H \rightarrow G' \]

Retrieve vertex:

\[ H \leftarrow G \rightarrow G' \]

- Identify generators

What are the fixed points of 2.0 \rightarrow 2.1

\[ H \leftarrow G \rightarrow G' \]

- Check if approach to complete transpose
- Check if approach to complete transpose is working
Prove by induction on $N$ that $g$ is a quasi-bp.

A "open-closed string map"

\begin{center}
\includegraphics[width=0.5\textwidth]{diagram}
\end{center}

Count pseudo-hol. sections of trivial $M$-bundle subject to
log'n boundary conditions

Degenerations vs. moduli:

\begin{center}
\includegraphics[width=0.5\textwidth]{diagram2}
\end{center}

$E_1$ is different basis in Hochschild differential.

Count holo. sections of the left fibration.

$s: CF_x(g) \to CF_x(M \circ \phi)$.

Gluing $\left[\text{Schwarz}\right]$ —
$s \circ \alpha$ counts section of the left fib. with log'n boundary conditions.
For $\mathbf{N}$, allow cut vertices to move around, due to their correct configurations.

Towards defining $\mathbf{N}$:

Cone $(\alpha)$ is a cone complex in the form of a $N$-adic hyperspace.

$N = 3$: 

\[ \{1, 2, 3\} \rightarrow \{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\} \]

In maps respect the set.

To each subset vertex (= subset of $\{1, 2, 3\}$) attach a summand in cone $(\alpha)$.

To $\emptyset = (\emptyset, \emptyset)$, attach $C_{F^*(\emptyset)}$.

For $I \in \{1, 2, 3\}$, attach summand in $C_{C^*(A, I)}$, involving the Lag's in $I$.

Define $\eta: C_{C^*} \rightarrow C_{F^*(\emptyset)}$.

Think of $I$ as a binary string.

Write down a planar rooted tree from this string.

- For each $0$, include a `bivalent vertex`
- For each $1$, do a `terminal vertex`.

$N = 3$ 000 leaf $\rightarrow$ root
110 \Rightarrow \quad \text{root}

Build a family of discs.

To each \quad \text{we attach the disc} \quad \text{in} \quad \text{out}

To each \quad \text{we attach} \quad 1 \quad \text{-param Family} \quad \text{in} \quad \text{out} \quad \text{preserve symmetry.}

Natural label labels for boundary guys.

Equip these discs with strip-like ends & glue:

\text{e.g.} \quad \text{in} \quad \text{out}

Finally, glue the outgoing prudence into \text{C-disc}.

picture.
Build by cut and paste a version of left fibration \( E \) whose cut values are the interior marked points of those discs. Define \( n \) by counting sectors w/ Lagrang boundary conditions, check that one gets a chain map/full homotopy of one.

**Inductive step:**

Build two S.E.S. \( 0 \to \mathbb{C}F_o(\beta) \to \mathbb{C}F_0(\tau^* \circ \beta) \to \mathbb{C}F_0(V_3, \beta V_1) \to 0 \)

\( \uparrow \text{Seidel} \)

\( \uparrow \text{truncation} \)

\( \uparrow \text{truncation} \)

\( 0 \to \mathbb{C}C_{x}(A_{N+1}, \beta_{N+1}) \to \mathbb{C}C_{x}(A_N, \beta_N) \to \text{Bar}^A_{A_{N-1}}(V_1, V_1) \to 0 \)

This should be something built from \( V_2, \ldots, V_{N-1} \).

Can interpret \( \Theta \) in terms of Seidel's twisting hom, which shows it to be a quasi-iso. (Uses \( n \geq 0 \) from of L.E.S.1.)
Finally — check that the spaces commute up to hom.

— apply 5-lemma to the L.E.S.

omitted for now.