Cyclic Homology as a wonderful corner of Chern Character.

- Chern Character & Cyclic Homology
- What structures does cyclic homology have? (Product on cyclic)
- Jones / Goodwillie

Loop space vs. Hochschild

Cyclic Homology Basics:

A algebra, unital, associative.

$$(C_\cdot(A,A), b')$$ bar resolution

$$(C_\cdot(A,A), b)$$ Hochschild chains

$${H^0}_\cdot(A) = H^0(A,A)$$

$${\text{Tor}}_\cdot^A(A,A)$$

$${C^\cdot(A,A), \beta}$$ Hochschild cochains

$${H^\cdot(A,A) = \text{Ext}^\cdot_A(A, A)}$$

$$(A^e \text{ is the dg where modules are } A \text{ bimodules})$$

A directed set of $V_i$ in a L.P.

E.g. sequence of curves on a surface.

\[\text{Ob: } N \text{ curves }\]

\[\text{hom}(x_i, x_j) = \begin{cases} 
CF^*(y_i, y_j), & \text{if } i = j \\
A & \text{if } i = j \\
0 & \text{if } i > j 
\end{cases}\]

\[\text{CF}(y_{i_0}, y_{i_1}) \otimes \text{CF}(y_{i_{r-1}}, y_{i_r}) \otimes \cdots \otimes \text{CF}(y_{i_0}, y_{i_1})\]

\[i_0 < \cdots < i_r\]
Get rid: this should be \((C^*(A,A), \beta)^\cdot\) (or \(C^*_\cdot(A,A^\cdot)\)).

Conjecture:

\[ H^\cdot F^*(M) \to H^\cdot (E) \]
\[ H^\cdot (A, A^\cdot) \]

Is there a deformation theory view to this?

Also, this should play a role in \(SH=HH\) picture.

Can do \(\text{Hom} (A, M^\cdot) \to \text{Ext}_A^\cdot (A, M^\cdot)\).

Define \(H^\cdot (A, M^\cdot) := \text{Tor}^A (M, A)\) \(\text{C.} (A, M^\cdot) := M \otimes A^\otimes\).

\(H^\cdot (A, N^\cdot) := \text{Ext}_A^\cdot (A, N^\cdot)\) for \(A\)-bimodules \(M, N\).

Operations:

\[ \cup : H^i (A, M)^\cdot \times H^j (A, N^\cdot) \to H^{i+j} (A, M \otimes_A N^\cdot) \]

i.e.: \((H^\cdot (A, A), \cup)\) algebra (graded comm.).

Rule: this is also the Yoneda product.

\([\cdot, \cdot] \text{ Gerstenhaber bracket = Poisson bracket on } (H^\cdot (A, A), \cup)\) (degree \(-1\)).
On $H_0(A, A)$, we have

$$B : C_n(A, A) \to C_{n+1}(A, A)$$

(recall original differential

$$b : C_n(A, A) \to C_{n-1}(A, A)$$

$H^k$ is a simplicial homology theory.

$$C_n(A, A) \to \tilde{C}_n(A, A) = A \ast \tilde{A} \otimes \tilde{A}, \quad \tilde{A} = A / k.$$ quasi-isomorphism (assuming unitality)

$$b(x \ast y) = xy - yx.$$

Claim: formula for $B$ is easier on $\tilde{C}_n$ than $C_n$.

On $\tilde{C}_n(A, A)$, $B$ is given by $(1 - t) \circ N$

$$t : \text{cyclic permutation on } A \otimes \cdots \otimes A$$

$$t(x \ast y) = (-1)^{n} y \ast x.$$

$N$ norm operator

$$\sum_{i=0}^{n} t^i$$

( clearly $t \circ N = 1 \otimes N$.)

$$S : x \ast y \mapsto 1 \otimes x \ast -y,$$

On $\tilde{C}_n(A, A)$, $B$ is just given by $\circ N$

From this morning's notes, "$S" is our $N$,

$$B(x) = 1 \otimes x,$$

$$B(x \ast y) = 1 \otimes x \ast y - \ast y \otimes x.$$
Mixed complexes: \( b^2 = 0, \quad B^2 = 0, \quad \text{ker } b \circ B + \text{im } B = 0. \)

like double complex, but we only have one grading.

Construct a bicomplex:

\[
\begin{array}{cccc}
C_3 & B & C_2 & B & C_1 \\
\downarrow & & \downarrow & & \downarrow \\
C_2 & B & C_1 & B & C_0 \\
\downarrow & & \downarrow & & \downarrow \\
C_1 & B & C_0 & 0 & 0 \\
\downarrow & & \downarrow & & \downarrow \\
C_0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\( \text{Betti } H_\ast(\mathcal{A}) \) (total complex)

\( C_\ast(\mathcal{A},\mathcal{A}) \otimes k[u,u^{-1}]/u \ k[u] \)

\( \text{differential } b + uB, \)

\( \text{like torsion coeffs., ordering } H. \ u/\text{coeff in } \Omega/\mathbb{Z} \).

(w.r.t. technical details, \( H_\ast(\mathcal{A},\mathcal{A}) = \text{diff. forms } \) for \( \mathcal{A} = \text{mg. gens. on others } k(\xi_1, \ldots, \xi_n) \).

\( H_n(\mathcal{A},\mathcal{A}) = \text{ n-forms } \)

Introduce \( \omega, \quad \text{deg } \omega = -2 \) (homologically)

\[
H_\ast(\Omega^\infty(\mathcal{M}),\Omega^\ast(\mathcal{M})) \cong \bigoplus_{n>0} \mathcal{O} \Omega^n(\mathcal{M}).
\]

So it's a myth that \( H_\ast(\Omega^\infty(\mathcal{M}),\Omega^\ast(\mathcal{M})) \) is \( \Omega^\ast(\mathcal{M}), \) it's related.

Ours. complex lives here:

\[
\begin{array}{c}
\text{HC.}(\mathcal{A}) \\
\text{"cyclic homology"}
\end{array}
\]

\[
\begin{array}{c}
\text{HC} \\
\text{"negative"}
\end{array}
\]

\[
\begin{array}{c}
\text{HP} \\
\text{"periodic"}
\end{array}
\]

But we could also do

\[
\begin{array}{c}
\text{direct sum vs. direct product}
\end{array}
\]

("issues").
For $A = \mathcal{C}_0 \mathbb{R} \llbracket X \rrbracket$

Spectral sequence: $H^i R \Rightarrow$ next page looks like:

$$
\Sigma^3 \xleftarrow{B} \Sigma^2

\Sigma^2 \xleftarrow{B} \Sigma^1

\Sigma \xleftarrow{B} \Sigma^0

\text{So}

HP_*(k[X]) = \bigoplus H^i R^*(X) \quad \text{("additive K-theory")}

+ \mathbb{Z}

Remark: $HC_*$ most related to alg. $K$-theory,

and $HC$ seems to be the difference between $HP_*$ and $HC_*$.  

Cyclic cohomology $HC^*(A) = (HC^*(A))^{\otimes}$

Q: Does $HP$ have Adams operations? A: if $A$ is commutative.

Digression: String topology.

Assume $Q$ smooth compact manifold, oriented

want $C_\ast (\Sigma^n Q)$, differential

multiplicativity comes from

$$
\Sigma^n Q \otimes \Sigma^n Q \to \Sigma^{2n} Q
$$

pass to chains

(actually $Aoo$ algebra, but we use nice loops, we're ok).

C_\ast (C_\ast (\Omega^n Q), C_\ast (\Omega^n Q)) is a chain model for homology of free loop space.
Why? Geometric realization —

G = BG 
BG = Q. Q is a classifying space for the monoid.

\[ \Omega Q \to PQ \]
\[ \Omega Q \to Q \]

\[ G \quad BG \quad EG \quad LBG \]
\[ \Omega Q \quad Q \quad PQ \quad LQ. \]

Two steps:

Use simplicial space structure.

Proof: Take these to some fibrations & model their chain complex.

Ref: Jones, Jour. Math, 1986. (only talks about cyclic)

\[ G \to G \times G \]
\[ BG \]
Next step: \( C^* (C_\omega (\Omega Q), C_\omega (\Omega Q)) \) is a chain model for homology of free loop space (as manifold, Pontryagin duality).

Have Yoneda product there.

1) Yoneda product agrees w/ product in string topology.

\[ A := C_\omega (\Omega Q) \]

\[ \rightarrow A \otimes_{C_\omega} A \rightarrow \text{ comes from forgetting base point} \]

\[ \downarrow \]

\[ C_\omega (\Omega Q) \rightarrow C_\omega (\Omega Q), \]

\[ \sigma_0 \circ \sigma_1 \]

\[ \downarrow \]

\[ (C_\omega, \sigma) \times_{\sigma_0} x_{\sigma_1} \rightarrow \Delta (Q) \]

\[ \text{moves basepoint along path, if just one of them} \]

compatible w/ differentials?

Part of diff in \( C_\omega : A \otimes A \rightarrow A \)

goes to \( 1 \sigma_0 \times_{\sigma_0} \sigma_1 \) \( \sigma_0 \times_{\sigma_0} \sigma_2 \普及 \text{natural guy} \)

How does this end up? \( \nabla \) pen down family of free loops

\( \text{maybe tangent lines by squarer function!} \)
\[ A^0 \otimes [2] \]

\[ \sigma_0 \otimes \cdots \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \]

**Case:** \( N^{\omega Y} G \) vs. \( \wedge BG \)

* involves Hochschild-like
* involves cyclic structure

\[ G \times G \]

\[ (g_0, g_1, \ldots, g_n) \]

\[ \text{with homology class } (t_0, t_1, \ldots, t_n) \]

\[ \Rightarrow \quad g_0 g_1 = g_0 \]

\[ \text{not quite well-defined by } \}

\[ \text{there's an up to conjugacy issue} \]

for higher groups,

\[ N^{\omega Y} G \xrightarrow{f} BG \]

* forgetting \( 0 \times \text{coord.} \)

\[ N^{\omega Y} G \rightarrow \wedge N^{\omega Y} G \xrightarrow{f} \wedge BG \]

* problem: There's a happy \( B\Sigma Q \rightarrow Q \), but

* may not be easy to realize.
Claim: cubical chains on Moore loops \( C_\omega(S^2 \omega Q) \) is a strict dga.

Teleman: Fillmore-Moore S.S.

\[
\begin{array}{c}
h^Y \rightarrow Y \\
\downarrow \\
X
\end{array}
\]

\( h^Y \) carries an action of \( S^2X \)

\( Y = \text{tot} \) space of associated bundle.

\( \text{hypo quotient} \) \( \text{left derived.} \)

Fact (w/o proof): \( C_\ast(Y/G) \cong C_\ast Y \otimes_{C_\ast G} k \)

\[
\begin{array}{c}
Y \xrightarrow{h} Z \\
\downarrow \\
X
\end{array}
\]

Fillmore-Moore:

\[ C_\ast(Y \times X^Z) \cong C_\ast(h^Y) \otimes_{C_\ast h^Z} C_\ast S^2X \]

Substitution:

\[
\begin{array}{c}
X \rightarrow X \times X \\
\uparrow \Delta \\
\downarrow \Delta \\
X \rightarrow X
\end{array}
\]

\( \text{so get:} \)

\[
\begin{array}{c}
\Omega(LX) \\
\Delta \rightarrow \Delta \\
X \rightarrow X
\end{array}
\]

1. \( X \times_{X \times X} X = LX \)

2. \( h_X : X \xrightarrow{\Delta} X \times X \xrightarrow{\omega} \Sigma X \)

\( \text{hypo fibre:} \)

\( \Sigma(X \times X) \).
Why is \( \Omega X \) actually \( \Sigma X ? \)

\[
\begin{align*}
\Omega X & \to \Sigma X \\
\downarrow & \quad \downarrow \Delta \\
* & \to \Sigma X
\end{align*}
\]

Claim:
Taking P.B., have to pass to spectra, can't do it here anymore.

Spheres & Whitehead dual picture.

Operations: don't want to necessarily look at transverse chains.

\( SH^* \) is a ring. What's the product structure?

Map: \( SH^* \to HH^*(C_{\text{W}}(T^*_q Q), C_{\text{W}}(T^*_q Q)) \)

\[
\begin{array}{c}
\downarrow \\
\left\langle Z \subset \Omega_q \mathcal{Q}, \text{so} \rightangle \\
HH^*(\mathcal{C}_{\text{x}}(\Omega_q \mathcal{Q}), \mathcal{C}_{\text{x}}(\Omega_q \mathcal{Q}))
\end{array}
\]

Product here is \( E_2 \), not \( E_3 \) like \( \Sigma \).

\( T^*_q \mathcal{Q} \). \( C_{\text{W}}^* \cong H^*(T^*_q Q) \otimes \mathbb{Z} < \text{Reeb chords of endpoints on } T^*_q Q > \).

*Elkholm
* Abbud-Bahri-Schwarz
* Fukaya-Seidel-Smith
What is the map $S H^* \rightarrow H H^*$

- Break at $\infty$: $x_0 \rightarrow D F(d \Phi)$.
- What about string breaking?
  - Can't have it here b/c we have a Hamiltonian turned on at extreme.

\[ \phi^*: \mathcal{A}^\infty \rightarrow \mathcal{A} \]

\[ \delta \Phi = \phi(\mathcal{I} \circ \phi \circ \mathcal{I}) + \mu(\mathcal{I} \circ \phi \circ \mathcal{I}) \]

Not quite hol. curves, hol. sections of a Riemann surface.
This is an iso:

Post. (Mohseni): Pass through Abbadandalo-Schwarz,
check commutivity of diagrams, or iso of \( H^{-}(\mathbb{L}, Q) \)
(maybe could do w/ energy filtration).

It need to be checked to show \( F \) can be enhanced to an \( \infty \) -multi

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Gottler:

Earlier, \( H_{\ast}(\mathcal{C}^{\infty}(U), \mathcal{C}^{\infty}(M)) \cong _{\Omega}(M) \).

\( \exists \) 'Adam's operations' on \( \mathcal{C}^{\infty}(U) \).

namely \( \psi_{k} \mid _{\mathcal{C}^{\infty}(U)} = k^{\frac{n}{2}}. \)

\( \psi_{k} \mid _{\mathcal{C}^{\infty}(U)} \) \( k \)-th by \( \mathcal{C}^{\infty}(U) \), \( k = 1, 2, 3, \ldots \)

Main thing: \( \psi_{k} \circ \psi_{l} = \psi_{kl} \).

At level of bar construction, this should be a left wrapping operation.

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Today

Burghelea - Vigué - Poirier.

something like this?

\[ x_{0} \otimes \cdots \otimes x_{n} \longrightarrow \delta \]

\[ x_{0} \otimes x_{1} \otimes \cdots \otimes x_{n} \otimes x_{1} \otimes \cdots \otimes x_{n} \otimes \delta \]

\[ C^{\infty}(M) \rightarrow \Omega^{\ast}(M). \]
by Chen:

\[ \mathcal{C}: (\Omega^*(M), \Omega^*(M)) \rightarrow \Omega^* (LM) \]

"Iterated Integral":

\[ \Omega^*(M^{n+2}) \]

pull back to \( \Delta^n \times LM \)

\[ \Delta^n \times LM \rightarrow M^{n+1} \]

in thus it is local, & this a so

Two operators on \( \Omega^* (LM) \) produced by \( B, \delta \).

Look at action of \( \text{Diff}(0,1) \) on \( \Omega^* (LM) \).

forms you get under \( \text{Diff}(0,1) \) action.

This is

Q: Only in commutative case.

But on other hand, it looks like we have relations between objects of different wrapping. Does these formulas below lift up?

A: Maybe yes for both chord picture.

Actually, no, b/c degree of \( \gamma, \delta \) don't grow linearly.

Is there some asymptotic leading order question?

In both cases we have things that want to be a power product, but fail.

So should be related!

And guys don't have smoothly.

If we believe \( H^* \), \( \text{Diff}(0,1) \) has a tensor product.

\[ \text{Ch}(X) \cong \bigoplus_{n \geq 0} \text{Fuk}(X^n) \]

(proved for semi-flat)

If \( X^n \) has a log torus fibration w/ a section then \( \text{Fuk}(X^n) \) has an \( n \)-cell manifold structure.
Idea:

"Add the Lag's fibre-wise is the tensor product.

\[ \text{HF}(L_1, L_1) \otimes \text{HF}(L_2, L_2) \rightarrow \text{HF}(L_1 \otimes L_2, L_1 \otimes L_2). \]

\[ (X_1^\Phi)^3 = T \text{ lag } \Phi. \]

\[ T = (x_1, x_2, x_3) \quad \pi(x_i) = \pi(x_j) \]

and \( x_1 + x_2 = x_3. \)

(Reverse symplectic fun on some factors.)

\( (L_1, L_2) \rightarrow L_1 \times L_2 \) is "composing \( L_1 \times L_2 \) with \( T. \)

Ref:

Alexander Subotic

If we have a Lag's vector space fibration, e.g. \( T^*Q, \rightarrow \)

\( \text{Fuk}(T^*Q) \) has a symmetric monoidal st.h. (so vector).

Zero section is id for \( \otimes \).

Changes fibration, \( \otimes \) might change, e.g. abelian varieties, (Fukai-Hofer)

\[ \otimes 1 \text{st class} \]

\[ C^\Phi(Q) \text{ is not quik} \]

Ref:

Peter: Hood, Jones, "K-theory Journal."

Q: Given module over Aco category, \( \mathcal{F} \) then character on cyclic homology? Yes, ref
Then: Let: Suppose $QH \cong HH$.

Given $L = M$, $ch(M) \in HH$, so $ch(M) \in QH$.

What is this? Answer: If $L$ actual log'n, it's $\mathcal{L}$. 

Problem: In mirror, $\mathcal{L}$ goes to minus vector, it's not a clean chance.

probably b/c $QH \cong H^{-1}(X^\vee)$ should be twisted.

I'm sure caution on the mirror.

$L$ should have notice # assigned to it, b/c

$[\mathcal{L}]$ goes to (here) forms in mirror.

which $L$?

(Fewer values of Adam's operation on torus fibrations??)

Getzler: In his example, Adam's operation were semi-simple

But will have unpoints here.

Hi! For $T^2$ in no ways,

![Diagram](image)

For filtration in $HC$, expect this to appear in $0$ and this higher, but there are too many structures giving rise to filtrations.

Call have a family of cohomology groups, maximal canon filtration around.