AJM Day 1 Talk 1, Bourgeois


\((x^{2n}, \omega)\) Liouville: \(\exists \nu, f, Z \text{ s.t. } L_Z \omega = \omega, \text{ with cylindrical end } \exists \text{cpt } \mathbb{R} \times \mathbb{R} \times X \text{ s.t. } x \times \mathbb{R}^{2n} \times \mathbb{R}^\infty \times Y \text{ and on } (0, \infty) \times Y.

\(Z = \frac{\partial}{\partial s}, \ L_Z \omega \mid_{Y} = \alpha, \text{ contact form on } Y.\)

\(z\) pointing outwards.

Ex: \(M^\star mfd; T^\star M, \omega, \text{ Liouville one form} \)

\((p, z) \text{ local coords } \rightarrow \lambda = p \cdot dp \omega = d\lambda = dp \cdot dp. \ Z = p \cdot \frac{\partial}{\partial p}. \)

\(X = D^\star M, Y = ST^\star M. \)

\((x^{2n}, \omega) \text{ Weinstein if } Z \text{ is gradient-like for a proper, bounded below, Morse fn. } H: X \rightarrow \mathbb{R}, \text{ i.e. } dH(\dot{z}) > S \lVert z \rVert^2 \text{ for some } S > 0.\)

(Just nee some } Z \text{ to exist; can change } Z, \text{ i.e. above, } Z \text{ for } T^\star X \text{ doesn't work.})

\text{crit. points of } H \text{ have Morse index } \leq n = \frac{1}{2} \dim X.\)
If crit. points have index < n: X is "subcritical" manifold.

- Stable manifolds of crit. points of H are isotropic.

Generalize to Liouville & Weinstein cobordisms:

W cobordism has $\partial W = \partial_+ W \cup \partial_- W$

$\partial_+ W$ - pointing inwards, pointing transverse, pointing outwards.

$\partial_- W$ - H: $\bar{W} \to \mathbb{R}$ is constant on

$\partial_- W \in \partial_+ W$ and $\partial_+ W$ (for Weinstein).

Interested in $W$ with all crit. points of $H$ of index $\leq n$.

$p_i : \mathbb{R}^n \to \mathbb{R}$

$p_i - p_k$ -> stable manifolds $L_i$, $l_k$ (no. ran by some that $L_i$'s don't intersect).

$\Lambda_i = \bar{L}_i \cap \partial_- \bar{W}$ Legendrian sphere in $\partial_- \bar{W}$.

Attaching $\bar{W}$ to $(X_0, \omega) \to (X, \omega)$ is a Legendrian surgery/handle attachment.

Framing for surgery along $\Lambda = \bigcup \Lambda_i$:

$\Lambda$ Legendrian $\subset \mathcal{E}X_0 = \mathcal{Y}_0$, contact $\to$ std. tubular with modelled on $J^1(\Lambda) = T^* \Lambda \times \mathbb{R}$

with contact form $dz - \partial \varphi (p, q)$ $\neq 0$ different $\mathcal{Y}^2$.

(open question)
II. Holomorphic curves

On $(X,\omega)$, choose a compatible a.c. structure $J$, i.e.
$J^2 = -I$, $\omega$ is $J$-invariant and $\omega(\cdot, J\cdot) > 0$.

$\omega(J, J\cdot) = \omega$.

Condition on cyl. end: $J(\ker \alpha) = \ker \alpha$,
$J_{25} = R_x = \text{Reeb field for } \alpha$, characterized
by $\{c(R_x) d\alpha = 0, \alpha(R_x) = 1. \}$.

$\mathcal{P}(\mathcal{Y})$ = \# closed orbits of $R_x$.

\(\mathcal{P}(\mathcal{Y})\) \(\geq\) closed orbits of $R_x$.

\(\mathcal{P}(\mathcal{Y})\) non-degenerate for generic $\alpha$ (i.e. linearize $R_x$ has
no eig = 1).

$X$ orbit $\mapsto \mathbb{Z}(x) \in \mathbb{Z}$.

(Need: A framing of contact str. on $\mathcal{X}$, depends on
$c_1$, other data).

Assume: $c_2(X) = 0$, canonical bundle trivialize.

$X$ closed orbit \(\leq\) bad: if $\mathcal{Y}$ is even multiple of $\mathcal{Y}$ and

\(\geq\) good: $c_2(\mathcal{X}) \neq c_2(\mathcal{Y})$ and 2.

\(\mathcal{P}(\mathcal{Y})\) \(\geq\) good ($\mathcal{Y}$).

\(\Sigma\), Riemann surface with boundary $\Sigma$,

punctures internal/boundary.

Holomorphic maps $f: \Sigma \to X$, i.e. $df \circ j = J \circ df$.

$+$ Lax boundary consisting, Lax's have cylinders $(0, \infty) \times \Sigma$. 

\(c_1\)
1. Asymptotic invariants of punctures:
   - Interval: $f$ converges to closed Reeb orbit at $\pm \infty$.
   - Boundary: $f$ converges to Reeb chord of $\Lambda$ at $\pm \infty$.

   (N.B. contribution from good orbit is $H_5^*(S') = \mathbb{Q}$
   contribution from bad orbits is $H^\infty_5(S', \text{unf})$
   by Möbius $(\cdot, \mathfrak{L})$ which is torsion & unwashes/$\mathbb{Q}$.
   
   (Claim: Morse theory of unstable, if unstable manifold not
   preserved by group action, throw it out).

2. Consider hol. curves in $W$ cased in $X_0$, i.e.:
   $X_0$
   - hol. maps $f: \{x_1, x_2, \ldots, x_n\} \to W$
   omitted for now
   s.t. $x_i$ converges to $x_i$ at $-\infty$

3. Hol. plane $h_i: \mathbb{C} \to X_0$ asymptotic at $+\infty$ to $x_i$

III. Hol. curve invariants of $X_0$

1. Linearized contact homology complex $\mathcal{C}(X_0)$ = module gen. by good closed Reeb orbit $\bar{x}_0 = \partial X_0$. 

differential $d_{CH}$ linear,

$$d_{CH} \gamma = \sum \# \cdot M_y(x, y') \cdot \gamma$$

$CZ(x') = CZ(x-1)$

(Ignoring signs now)

Prop: $d^2_{CH} = 0$, $CH(X_0)$ :=

$H(C_{CH}(X_0), d_{CH})$ is mvt. of $X_0$ up to sign.

2 Reduced Symplectic Homology

complex $SH^+(X_0) = CH \oplus \hat{CH}(X_0)$

non-equivalent:

$CH(X_0)$ $\hat{CH}(X_0)$ [17]

(On each orbit $Y$, choose perfect Morse (en, perturb to get two generators).

differential $d_{SH^+}$ linear:

$$d_{SH^+} = \begin{pmatrix} d_{CH} & 0 \\ 0 & d_{\hat{CH}} \end{pmatrix}$$

with

$d_{SH^+} = d_{CH}$, $d_{\hat{SH^+}} = d_{\hat{CH}}$ and

$$s \gamma = \sum \# \cdot \hat{M}_x(y, y') / IR \cdot \gamma$$

$(CZ(x') = CZ(x) - 2)$
2.2 generator.

So we make a bigon complex of local orbit.

Here we don't throw away bad orbit.

R admits an x such that x, y, z = 3 points P, P, Q on x.

R admit configurations of good cyclic.

x 

special line R x 315.

(yw + xz) k x 4 x
(3) (Full) symplectic homology

complex \( SH(X_0) = \bigoplus_{m \in \mathbb{Z}} \bigoplus_{n \in \mathbb{Z}} \text{Morse} (-H)[n] \)

differential
\[
\partial_{SH} = \begin{pmatrix}
\partial_{SH} & 0 \\
0 & \partial_{\text{Morse}}
\end{pmatrix},
\]

\[\theta^\wedge = 0\]
\[\theta^\vee = \sum_{\|\gamma\|_1 = 1} \mathcal{M}_\gamma(X, \rho) \cdot \rho\]

marked pt. \( a + 0 \in \mathbb{C} \).

Prop: \( d_{SH}^2 = 0 \), \( SH(X_0) \)
\[H(SH(X_0), d_{SH}) \text{ is invl. of } X_0 \text{ up to sympl.}\]

\[H^{-*+*}(X_0) \to SH(X_0)\]

\(\check{\triangleright} \) 

\( SH(X_0) \)
Q: Do these series converge?

Rule: Counting hot, anger

Let $f(x) = \text{sum} \rightarrow (x)$ and $g(x) = \text{chairs} \rightarrow (x)$. What then?