Randomization
-can be used to improve algorithms
Example: Testing for prime numbers using coin flips

Sorting
Quick Sort
1. choose first element and treat it as the pivot
2. make two groups
   - one greater than the pivot
   - one less than the pivot
3. recurse with each new group

Perfect split:
\[ T(32) = 2T(16) + 32 \]

Less Perfect split:
\[ T(32) = T(16) + T(24) + 32 \]

\[ T(32) = 1 + T(31) + 31 \]
\[ \sqrt{2} (n^2) \]
How can we better choose the pivot?

Randomness

\[
\text{Coin } \in \{\text{H, T}\} \quad \{0, 1\}
\]

\[
\text{Probability } \left[ \text{Coin } = "H" \right] = \frac{1}{2} = \Pr \left[ \text{Coin } = "T" \right]
\]

("Perfect Coin")

\[
\text{K-dice } \{0, 1, \ldots, k - 1\}
\]

\[
\Pr \left[ \text{K-dice } = i \right] = \frac{1}{k} \quad \text{(uniform dice)}
\]

- all outcomes have equal probability

n-elements \underline{2-dice 3-dice \ldots n-dice}

Selection Problem

\[
\text{INPUT: } A[1], A[2], \ldots, A[n] \quad \text{ - array of #s}
\]

\[
k \quad \text{- integer}
\]

\[
\text{OUTPUT: } k^{th} \text{ smallest element of } A[1]
\]

How quickly can we solve this problem using randomness?
Suppose array has 1 - 32 elements looking for 24

- randomly choose 60,
- use as pivot and sort into 2 groups, \(<60\) and \(\geq 60\)
- since 24 is \(\geq 60\), throw away \(<60\) group
- (recursively, using \(\geq 60\) group
 \[ T(32) = T(26) + 32 \]

Randomness useful in selecting pivot

\[ E \left( T_{Rs}(n) \right) = O(n) \]

\[ E \left( Iteration(n) \right) = O(\log(n)) \]

- smallest \(\frac{1}{4}\) \(\frac{1}{4}\) largest

\(25\%\) \(\frac{1}{2}\) \(75\%\)

If \(x\) is random from \([1, n]\) uniformly:

\[ \Pr \left[ \frac{n}{4} \leq x \leq \frac{3n}{4} \right] = \frac{1}{2} \]
\[ E \left( T_{\text{RS}}(n) \right) = \frac{1}{2} E \left( T_{\text{RS}} \left( \frac{3}{4} n \right) \right) + \frac{1}{2} E \left( T_{\text{RS}}(n) \right) + n \]

\[
E \left( T_{\text{RS}}(n) \right) \leq E \left( T_{\text{RS}} \left( \frac{3}{4} n \right) \right) + 2n \\
\downarrow \\
E \left( T_{\text{RS}} \left( \frac{3}{4} n \right)^2 \right) + 2 \cdot \frac{3}{4} n + 2n \\
\downarrow \\
= 1 + \cdots + 2n \left( \frac{3}{4} \right)^3 + 2n \left( \frac{3}{4} \right)^2 + 2n \left( \frac{3}{4} \right) + 2n \\
= \Theta(n) \\
E \left( T_{\text{QSort}}(n) \right) = \Theta(n \log n) \\

\text{Birthday Paradox} -

suppose: a year has n days 
m people with random birthdays

\[
Pr[\text{two of them have the same birthday}] = \begin{cases} 
0 & \text{if } m = 1 \\
\frac{1}{n} & \text{if } m = 2 \\
\frac{1}{n^2} & \text{if } m = 3
\end{cases}
\]
\# 3-way collisions + # 2-way collisions
\[
\frac{n}{n^3} + \frac{3n(n-1)}{n^3}
\]
= number of 3-way collisions

\[m = 100\]
\[\binom{m}{2} = \frac{m(m-1)}{2} \text{ pairs} \]
\[\Pr[m, n] \geq \frac{m(m-1)}{2} \cdot n \cdot n^{m-2} \]
\[
\geq \frac{n^m}{n^m}
\]
= \[\frac{m(m-1)}{2n}\]