Outline

1. Examples of Mechanism Design Problems
2. The General Mechanism Design Problem
3. The Revelation Principle and Incentive Compatibility
4. Mechanisms with Money: The Quasilinear Utility Model
5. Maximizing Welfare: The VCG Mechanism
6. Maximizing Revenue
   - The Setup: Single-Parameter Bayesian Revenue Maximization
   - Characterization of BIC
   - Myerson’s Revenue-Optimal Auction
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Single-item Allocation

- $n$ players
- Player $i$’s private data (type): $v_i \in \mathbb{R}_+$
- Outcome: choice of a winning player, and payment from each player
- Utility of a player for an outcome is his value for the outcome if he wins, less payment

Objectives: Revenue, welfare.
Single-item Allocation

First Price Auction

1. Collect bids
2. Give to highest bidder
3. Charge him his bid
Single-item Allocation

Second-price (Vickrey) Auction

1. Collect bids
2. Give to highest bidder
3. Charge second highest bid
Example: Combinatorial Allocation

- \( n \) players, \( m \) items.
- Private valuation \( v_i : \text{set of items} \to \mathbb{R} \):
  - \( v_i(S) \) is player \( i \)'s value for bundle \( S \).
Example: Combinatorial Allocation

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- Private valuation $v_i: \text{set of items} \rightarrow \mathbb{R}$.
  - $v_i(S)$ is player $i$’s value for bundle $S$.

Goal

Partition items into sets $S_1, S_2, \ldots, S_n$ to maximize welfare:

$v_1(S_1) + v_2(S_2) + \ldots v_n(S_n)$
Example: Public Project

- $n$ players
- Player $i$’s private data (type): $v_i \in \mathbb{R}_+$
- Outcome: choice of whether or not to build, and payment from each player covering the cost of the project if built
- Utility of a player for an outcome is his value for the project if built, less his payment

Goal: Build if sum of values exceeds cost (maximize welfare), or maximize revenue
Example: Voting

- $n$ players
- $m$ candidates
- Player $i$’s private data (type): total preference order on candidates
- Outcome: choice of winning candidate

Goal: ??
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Mechanism Design Setting (Prior free)

Given by a tuple \((N, \mathcal{X}, T, u)\), where

- \(N\) is a finite set of players. Denote \(n = |N|\) and \(N = \{1, \ldots, n\}\).
- \(\mathcal{X}\) is a set of outcomes.
- \(T = T_1 \times \ldots T_n\), where \(T_i\) is the set of types of player \(i\). Each \(\vec{t} = (t_1, \ldots, t_n) \in T\) is called an type profile.
- \(u = (u_1, \ldots u_n)\), where \(u_i : T_i \times \mathcal{X} \to \mathbb{R}\) is the utility function of player \(i\).
Mechanism Design Setting (Prior free)

Given by a tuple \((N, \mathcal{X}, T, u)\), where

- \(N\) is a finite set of **players**. Denote \(n = |N|\) and \(N = \{1, \ldots, n\}\).
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- \(u = (u_1, \ldots, u_n)\), where \(u_i : T_i \times \mathcal{X} \to \mathbb{R}\) is the **utility function** of player \(i\).

In a **Bayesian** setting, supplement with common prior \(\mathcal{D}\) over \(T\).
Mechanism Design Setting (Prior free)

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- \(u = (u_1, \ldots u_n)\), where \(u_i : T_i \times X \to \mathbb{R}\) is the utility function of player \(i\).

In a Bayesian setting, supplement with common prior \(\mathcal{D}\) over \(T\)

Example: Single-item Allocation

- Outcome: choice \(x \in \{e_1, \ldots, e_n\}\) of winning player, and payment \(p_1, \ldots, p_n\) from each
- Type of player \(i\): value \(v_i \in \mathbb{R}_+\).
- \(u_i(v_i, x) = v_i x_i - p_i\).
A principal wants to communicate with players and aggregate their private data (types) into a choice of outcome. Such aggregation captured by

A social choice function $f : T \rightarrow X$ is a map from type profiles to outcomes.
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A social choice function $f : T \rightarrow \mathcal{X}$ is a map from type profiles to outcomes.

Choosing a Social Choice Function

- A particular social choice function in mind (e.g. majority voting, utilitarian allocation of a single item, etc).
- An objective function $o : T \times \mathcal{X} \rightarrow \mathbb{R}$, and want $f(T)$ to (approximately) maximize $o(T, f(T))$
  - Either worst case over $T$ (Prior-free) or in expectation (Bayesian)

Example: Single-item Allocation

- Welfare objective: $\text{welfare}(v, (x, p)) = \sum_i v_i x_i$
- Revenue objective: $\text{revenue}(v, (x, p)) = \sum_i p_i$
To perform aggregation, principal runs protocol called a mechanism.

A mechanism is a pair \((A, g)\), where

- \(A = A_1 \times \cdots A_n\), where \(A_i\) is the set of possible actions (think messages, or bids) of player \(i\) in the protocol. \(A\) is the set of action profiles.
- \(g : A \to X\) is an outcome function.
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1. \(A = A_1 \times \ldots \times A_n\), where \(A_i\) is the set of possible actions (think messages, or bids) of player \(i\) in the protocol. \(A\) is the set of action profiles.
2. \(g: A \rightarrow X\) is an outcome function

The resulting game of mechanism design is a game of incomplete information where when players play \(a \in A\), player \(i\)'s utility is \(u_i(t_i, g(a))\) when his type is \(t_i\).

Example: First price auction

1. \(A_i = \mathbb{R}\)
2. \(g(b_1, \ldots, b_n) = (x, p)\) where \(x_{i^*} = 1\), \(p_{i^*} = b_{i^*}\) for \(i^* = \arg\max_i b_i\), and \(x_i = p_i = 0\) for \(i \neq i^*\).
We say a mechanism \((A, g)\) implements social choice function \(f : T \rightarrow X\) in dominant-strategy/Bayes-Nash equilibrium if there is a strategy profile \(s = (s_1, \ldots, s_n)\) with \(s_i : T_i \rightarrow A_i\) such that

- \(s_i : T_i \rightarrow A_i\) is a dominant-strategy/Bayes-Nash equilibrium in the resulting incomplete information game
- \(g(s_1(t_1), s_2(t_2), \ldots, s_n(t_n)) = f(t_1, t_2, \ldots, t_n)\) for all \(t \in T\)

**Example: First price, two players, i.i.d \(U[0, 1]\)**

Implements in BNE the following social choice function: give the item to the player with the highest value and charges him half his value.

**Example: Vickrey Auction**

Implements in DSE the following social choice function: give the item to the player with the highest value and charges him the second highest value.
Given a notion of a “good” social choice function from $T$ to $X$, find
- A mechanism
  - An action space $A = (A_1, \ldots, A_n)$,
  - an outcome function $g : A \rightarrow X$,
- an equilibrium $(s_1, \ldots, s_n)$ of the resulting game of mechanism design
such that the social choice function $f(t_1, \ldots, t_n) = g(s_1(t_1), \ldots, s_n(t_n))$ is “good.”
The Task of Mechanism Design

Task of Mechanism Design (Take 1)
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Problem
This seems like a complicated, multivariate search problem.
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Luckily

The revelation principle reduces the search space to just $g : T \to X$.
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Incentive-Compatibility

Direct Revelation

A mechanism \((A, g)\) is a **direct revelation mechanism** if \(A_i = T_i\) for all \(i\).

i.e. in a direct revelation mechanism, players simultaneously report types (not necessarily truthfully) to the mechanism. Such mechanisms can simply be described via the function \(g : T \rightarrow X\).

Incentive-Compatibility

A direct-revelation mechanism is dominant-strategy/Bayesian **incentive-compatible** (aka **truthful**) if the truth-telling is a dominant-strategy/Bayes-Nash equilibrium in the resulting incomplete-information game.

Note: A direct revelation incentive-compatible mechanism implements its outcome function \(g : T \rightarrow X\), by definition.

The social choice function IS the mechanism!!
Examples

Vickrey Auction
Direct revelation mechanism, dominant-strategy incentive-compatible.

First Price Auction
Direct revelation mechanism, not Bayesian incentive compatible.

Example: Posted price
The auction that simply posts a fixed price to players in sequence until one accepts is not direct revelation.
Revelation Principle

If there is a mechanism implementing social choice function $f$ in dominant-strategy/Bayes-Nash equilibrium, then there is a direct revelation, dominant-strategy/Bayesian incentive-compatible mechanism implementing $f$. This simplifies the task of mechanism design.

Task of Mechanism Design (Take 2)

Given a notion of a "good" social choice function from $T$ to $X$, find such a function $f: T \rightarrow X$ such that truth-telling is an equilibrium in the following mechanism:

- Solicit reports $\tilde{t}_i \in T_i$ from each player $i$ (simultaneous, sealed bid)
- Choose outcome $f(\tilde{t}_1, \ldots, \tilde{t}_n)$
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- Choose outcome $f(\tilde{t}_1, \ldots, \tilde{t}_n)$
Example

2 players, with values i.i.d uniform from $[0, 1]$, facing the first-price auction.

**First-price Auction**

1. Solicit bids $b_1, b_2$
2. Give item to highest bidder, charging him his bid

**Recall**

The strategies where each player reports half their value are in BNE. In other words, when player 1 knows his value $v_1$, and faces player 2 who is bidding uniformly from $[0, 1/2]$, he maximizes his expected utility $(v_1 - b_1).2b_1$ by bidding $b_1 = v_1/2$. And vice versa.
Example

2 players, with values i.i.d uniform from $[0, 1]$, facing the first-price auction.

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Therefore . . .

the first price auction implements in BNE the social choice function which gives the item to the highest bidder, and charges him half his bid.
Example

Modified First-price Auction

1. Solicit bids $b_1, b_2$
2. Give item to highest bidder, charging him half his bid
   - Equivalently, simulate a first price auction where bidders bid $b_1/2, b_2/2$

Claim

Truth-telling is a BNE in the modified first-price auction.

Therefore, the modified auction implements the same social-choice function in equilibrium, but is truthful.
Example

Modified First-price Auction

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Claim

Truth-telling is a BNE in the modified first-price auction.

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Proof

Assume player 2 bids truthfully. Player 1 faces a (simulated) first price auction where his own bid is halved before participating, and player 2 bids uniformly from $[0, 1/2]$. To respond optimally in the simulation, he bids $b_1 = v_1$ and lets the mechanism halve his bid on his behalf.
Consider mechanism \((A, g)\), with BNE strategies \(s_i : T_i \rightarrow A_i\).

- Implements \(f(t_1, \ldots, t_n) = g(s_1(t_1), \ldots, s_n(t_n))\) in BNE
- For all \(i\) and \(t_i\), action \(s_i(t_i)\) maximizes player \(i\)'s expected utility when other players are playing \(s_{-i}(t_{-i})\) for \(t_{-i} \sim D|t_i\).
Proof (Bayesian Setting)

Consider mechanism \((A, g)\), with BNE strategies \(s_i : T_i \to A_i\).

- **Implements** \(f(t_1, \ldots, t_n) = g(s_1(t_1), \ldots, s_n(t_n))\) in BNE
- **For all** \(i\) and \(t_i\), action \(s_i(t_i)\) maximizes player \(i\)'s expected utility when other players are playing \(s_{-i}(t_{-i})\) for \(t_{-i} \sim D|t_i\).

**Modified Mechanism**

1. Solicit reported types \(\tilde{t}_1, \ldots, \tilde{t}_n\)
2. Choose outcome \(f(\tilde{t}_1, \ldots, \tilde{t}_n) = g(s_1(\tilde{t}_1), \ldots, s_n(\tilde{t}_n))\)
   - Equivalently, simulate \((A, g)\) when players play \(s_i(t_i)\)
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Consider mechanism \((A, g)\), with BNE strategies \(s_i : T_i \rightarrow A_i\).

- Implements \(f(t_1, \ldots, t_n) = g(s_1(t_1), \ldots, s_n(t_n))\) in BNE

- For all \(i\) and \(t_i\), action \(s_i(t_i)\) maximizes player \(i\)'s expected utility when other players are playing \(s_{-i}(t_{-i})\) for \(t_{-i} \sim D|t_i\).

Modified Mechanism

1. Solicit reported types \(\tilde{t}_1, \ldots, \tilde{t}_n\)
2. Choose outcome \(f(\tilde{t}_1, \ldots, \tilde{t}_n) = g(s_1(\tilde{t}_1), \ldots, s_n(\tilde{t}_n))\)
   - Equivalently, simulate \((A, g)\) when players play \(s_i(t_i)\)

- Assume all players other than \(i\) report truthfully

- When \(i\)'s type is \(t_i\), other players playing \(s_{-i}(t_{-i})\) for \(t_{-i} \sim D|t_i\) in simulated mechanism

- As stated above, his best response in simulation is \(s_i(t_i)\).

- Mechanism transforms his bid by applying \(s_i\), so best to bid \(t_i\).
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To make much of modern mechanism design possible, we assume that

- The set of outcomes has a particular structure: every outcome includes a payment to or from each player.
- Player utilities vary linearly with their payment.

Examples: Single-item allocation, public project,
Non-examples: Single-item allocation without money, voting.
Quasilinear Utilities

The Quasi-linear Setting

Formally, \( \mathcal{X} = \Omega \times \mathbb{R}^n \).

- \( \Omega \) is the set of allocations
- For \((\omega, p_1, \ldots, p_n) \in \mathcal{X}\), \(p_i\) is the payment from (or to) player \(i\).

and player \(i\)'s utility function \(u_i : T_i \times \mathcal{X} \rightarrow \mathbb{R}\) takes the following form

\[
u_i(t_i, (\omega, p_1, \ldots, p_n)) = v_i(t_i, \omega) - p_i\]

for some valuation function \(v_i : T_i \times \Omega \rightarrow \mathbb{R}\).

We say players have quasilinear utilities.

Example: Single-item Allocation

- \( \Omega = \{e_1, \ldots, e_n\} \)
- \( u_i(t_i, (\omega, p_1, \ldots, p_n)) = t_i \omega_i - p_i \)
Further simplification

Recall that, using the revelation principle, we got

Task of Mechanism Design (Take 2)

Given a notion of a “good” social choice function from $T$ to $X$, find such a function $f : T \rightarrow X$ such that truth-telling is an equilibrium in the following mechanism:

- Solicit reports $\tilde{t}_i \in T_i$ from each player $i$ (simultaneous, sealed bid)
- Choose outcome $f(\tilde{t}_1, \ldots, \tilde{t}_n)$
Further simplification

In quasilinear settings this breaks down further

**Task of Mechanism Design in Quasilinear settings**

Find a “good” allocation rule \( f : T \to \Omega \) and payment rule \( p : T \to \mathbb{R}^n \) such that the following mechanism is incentive-compatible:

- Solicit reports \( \tilde{t}_i \in T_i \) from each player \( i \) (simultaneous, sealed bid)
- Choose allocation \( f(\tilde{t}) \)
- Charge player \( i \) payment \( p_i(\tilde{t}) \)

We think of the mechanism as the pair \((f, p)\).
Sometimes, we abuse notation and think of type \( t_i \) directly as the valuation \( v_i : \Omega \to \mathbb{R} \).
Incentive-Compatibility

Incentive compatibility can be stated simply now

Incentive-compatibility (Dominant Strategy)

A mechanism \((f, p)\) is dominant-strategy truthful if, for every player \(i\), true type \(t_i\), possible mis-report \(\tilde{t}_i\), and reported types \(t_{\not i}\) of the others, we have

\[
v_i(t_i, f(t)) - p_i(t) \geq v_i(t_i, f(\tilde{t}_i, t_{\not i})) - p_i(\tilde{t}_i, t_{\not i})
\]

If \((f, p)\) randomized, add expectation signs.
Incentive-Compatibility

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### Incentive-compatibility (Dominant Strategy)

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If \((f, p)\) randomized, add expectation signs.

### Incentive-compatibility (Bayesian)

A mechanism \((f, p)\) is Bayesian incentive compatible if, for every player \(i\), true type \(t_i\), possible mis-report \(\tilde{t}_i\), the following holds in expectation over \(t_{-i} \sim D|t_i\)

\[
E[v_i(t_i, f(t)) - p_i(t)] \geq E[v_i(t_i, f(\tilde{t}_i, t_{-i})) - p_i(\tilde{t}_i, t_{-i})]
\]
Examples

### Vickrey Auction
- Allocation rule maps $b_1, \ldots, b_n$ to $e_{i^*}$ for $i^* = \arg\max_i b_i$
- Payment rule maps $b_1, \ldots, b_n$ to $p_1, \ldots, p_n$ where $p_{i^*} = b_2$, and $p_i = 0$ for $i \neq i^*$.

Dominant-strategy truthful.

### First Price Auction
- Allocation rule maps $b_1, \ldots, b_n$ to $e_{i^*}$ for $i^* = \arg\max_i b_i$
- Payment rule maps $b_1, \ldots, b_n$ to $p_1, \ldots, p_n$ where $p_{i^*} = b_1$, and $p_i = 0$ for $i \neq i^*$.

For two players i.i.d $U[0, 1]$, players bidding half their value is a BNE. Not Bayesian incentive compatible.
Modified First Price Auction

- Allocation rule maps $b_1, \ldots, b_n$ to $e_{i^*}$ for $i^* = \arg\max_i b_i$

- Payment rule maps $b_1, \ldots, b_n$ to $p_1, \ldots, p_n$ where $p_{i^*} = b_{(1)}/2$, and $p_i = 0$ for $i \neq i^*$.

For two players i.i.d $U[0, 1]$, Bayesian incentive compatible.
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In quasilinear setting, a simple mechanism is DSE and maximizes the social welfare $\sum_i v_i(\omega)$.
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**Vickrey Clarke Groves (VCG) Mechanism**

1. Solicit type $v_i$ from each player $i$
2. Choose allocation $\omega^* \in \arg\max_{\omega \in \Omega} \sum_i v_i(\omega)$
3. Charge each player $i$ payment $p_i(v) = h_i(v_{-i}) - \sum_{j \neq i} v_j(\omega^*)$

- Allocation rule maximizes welfare exactly over $\Omega$
- Player $i$ is paid the reported value of others for the chosen allocation, less a pivot term $h_i(v_{-i})$ independent of his own bid.
In quasilinear setting, a simple mechanism is DSE and maximizes the social welfare $\sum_i v_i(\omega)$

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- Allocation rule maximizes welfare exactly over $\Omega$
- Player $i$ is paid the reported value of others for the chosen allocation, less a pivot term $h_i(v_{-i})$ independent of his own bid.
- In most cases, the “right” pivot term is $\max_{\omega \in \Omega} \sum_{j \neq i} v_j(\omega)$
  - Payment $p_i(v)$ is player $i$’s externality
  - $0 \leq p_i(v) \leq v_i(\omega^*)$
Theorem

VCG is dominant-strategy truthful.
Proof

- Fix reports $v_{-i}$ of players other than $i$.
- Assume player $i$’s true valuation is $v_i$
- Player $i$’s utility when reporting $\hat{v}_i$ is given by

$$u_i(\hat{v}_i) = v_i(\omega^*) + \sum_{j \neq i} v_j(\omega^*) - h_i(v_{-i}),$$

where $\omega^* \in \arg\max_{\omega \in \Omega} \left(\hat{v}_i(\omega) + \sum_{j \neq i} v_j(\omega)\right)$

- Since the pivot term is independent of player $i$’s bid, maximizing $u_i(\hat{v}_i)$ is equivalent to maximizing

$$v_i(\omega^*) + \sum_{j \neq i} v_j(\omega^*)$$

- Setting $\hat{v}_i = v_i$ then maximizes the above expression.
  - Interpretation: allow the mechanism to optimize player $i$’s utility on his behalf
Example: Single-item Allocation

- Welfare maximizing outcome: Allocate to player with highest value

Externality of \( i \): second-highest value if \( i \) wins, 0 otherwise.

VCG is the second-price (Vickrey) auction in the special case of single-item allocation.
Example: Single-item Allocation

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Well understood in the case of **single-parameter problems**

**Single-parameter problem (informally)**

- There is a single homogenous resource.
- Constraints on how much of the resource each player can get
- Each player’s type is his “value (or cost) per unit resource.”
Well understood in the case of single-parameter problems

Single-parameter problem (informally)
- There is a single homogenous resource.
- Constraints on how much of the resource each player can get
- Each player’s type is his “value (or cost) per unit resource.”

Canonical example: single-item allocation
- Resource: one unit of item
- Outcomes $\Omega$: vectors $(x_1, \ldots, x_n)$ with $x_i \geq 0$ and $\sum_i x_i \leq 1$
  - $x_i$ is probability player $i$ gets item
- Player $i$’s type is $v_i \geq 0$ (value for item)
  - $u_i(x, p) = v_i x_i - p_i$
Maximizing Revenue

Makes most sense in Bayesian setting with independent types (prior $\mathcal{F} = \mathcal{F}_1 \times \ldots \times \mathcal{F}_n$ on $(v_1, \ldots, v_n)$)

Bayesian Revenue Maximization (Single Parameter)

Given prior $\mathcal{F}$ on type profiles $T \subseteq \mathbb{R}^n$, find allocation rule $x : T \rightarrow \Omega$ (recall $\Omega \subseteq \mathbb{R}^n$) and payment rules $p : T \rightarrow \mathbb{R}^n$ such that

- $(x, p)$ is a BIC direct revelation mechanism
- Bidding $b_i = v_i$ maximizes $\mathbb{E}_{v \sim \mathcal{F}} [vix_i(b_i, v_i) - p_i(b_i, v_i)]$
- $Rev(x, p) = \mathbb{E}_{v \sim \mathcal{F}} \sum_i p_i(v)$ is as large as possible.
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Myerson characterized the optimal solution for single-item auctions, and it generalizes easily to single-parameter environments

- Think of single-item auctions in upcoming discussion
Stages of a Bayesian Game

Stages of a Bayesian game of mechanism design:

- **Ex-ante**: Before players learn their types
- **Interim**: A player learns his type, but not the types of others.
- **Ex-post**: All player types are revealed.

Interim stage is when players make decisions.

The interim allocation rule for player \( i \) tells us what the probability (expected amount of resource) is as a function of player \( i \)'s bid, in expectation over other player's truthful reports.

\[
x_i(b_i) = \mathbb{E}_{v_i \sim F_i}[x_i(b_i, v_{-i})]
\]

Similarly, the interim payment rule.

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p_i(b_i) = \mathbb{E}_{v_i \sim F_i}[p_i(b_i, v_{-i})]
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Maximizing Revenue 25/34
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Assume two players drawn independently from $U[0, 1]$.

**Vickrey Auction**

- $\bar{x}_i(v_i) = v_i$
- $\bar{p}_i(v_i) = v_i^2 / 2$. 

First Price Auction

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Myerson’s Monotonicity Lemma

Consider a mechanism for a single-parameter problem in a Bayesian setting where player values are independent. A direct-revelation mechanism with interim allocation rule $x$ and payment rule $p$ is BIC if and only if for each player $i$:

- $x_i(b_i)$ is a monotone non-decreasing function of $b_i$
- $p_i(b_i)$ is an integral of $b_i \, dx_i$. Specifically, when $p_i(0) = 0$ then

$$p_i(b_i) = b_i \cdot x_i(b_i) - \int_{b=0}^{b_i} x_i(b) \, db$$
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The higher a player bids, the higher the probability of winning.

For each additional sliver $\epsilon$ of winning probability, pays at a rate equal to the minimum bid needed to acquire that sliver.

- Recall: second price auction
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See readings for proof of Myerson’s monotonicity Lemma
### Corollaries of Myerson’s Monotonicity Lemma

<table>
<thead>
<tr>
<th>Corollaries</th>
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<tbody>
<tr>
<td>Interim allocation rule uniquely determines interim payment rule.</td>
</tr>
<tr>
<td>Expected revenue depends only on the allocation rule</td>
</tr>
</tbody>
</table>

### Theorem (Revenue Equivalence)

*Any two auctions with the same interim allocation rule in BNE have the same expected revenue in the same BNE.*
Define the virtual value of player $i$ as a function of his value $v_i$

$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

Lemma (Myerson’s Virtual Welfare Lemma)

Consider a BIC mechanism $M$ with interim allocation rule $x$ and payment rule $p$, and assume that $p_i(0) = 0$ for all $i$. The expected revenue of $M$ is equal to the expected virtual welfare served.

$$\sum_i E_{v_i \sim F_i}[\phi(v_i) x(v_i)]$$

In single-item auction, this is the expected virtual value of the winning bidder.
\[
\mathbf{E}_{v \sim F_i} \left[ p_i(v) \right] = \int_v \left[ vx_i(v) - \int_{b=0}^v x_i(b) \, db \right] f_i(v) \, dv \\
= \int_v vx_i(v) f_i(v) \, dv - \int_v \int_{b \leq v} x_i(b) f_i(v) \, db \, dv \\
= \int_v vx_i(v) f_i(v) \, dv - \int_b x_i(b) \int_{v \geq b} f_i(v) \, dv \, db \\
= \int_v vx_i(v) f_i(v) \, dv - \int_b x_i(b)(1 - F_i(b)) \, db \\
= \int_v \left[ vx_i(v) f_i(v) - x_i(v)(1 - F_i(v)) \right] \, dv \\
= \int_v f_i(v) x_i(v) \left[ v - \frac{1 - F_i(v)}{f_i(v)} \right] \, dv = \int_v f_i(v) x_i(v) \phi_i(v_i) \, dv
\]
1. Solicit player values

2. If at least one player has nonnegative virtual value, then give the item to the player $i$ with the highest virtual value $\phi_i(v_i) \geq 0$. Otherwise, nobody gets the item.

3. Charge the minimum bid needed to win $\phi_i^{-1}(\max(0, \max_{j \neq i} \phi_j(v_j)))$.
   
   Check: satisfies Myerson’s condition on interim payment.
## Myerson’s Revenue-Optimal Auction

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## Observations

- The allocation rule maximizes virtual welfare point-wise
- Therefore, it maximizes expected virtual welfare over all allocation rules.
- By Myerson’s virtual welfare Lemma, revenue is at least that of any BIC mechanism (since any BIC mechanism’s revenue is equal to expected virtual welfare).
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Are we done?
A Wrinkle

Not really... What if the allocation rule of the mechanism we just defined is non-monotone? It would still have revenue at least that of the optimal BIC mechanism if players happened to report truthfully, but it wouldn’t be truthful itself.
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Fortunately

Virtual welfare maximization is monotone when the distributions are regular!!

\[ \phi_i(v) = v - \frac{1-F_i(v)}{f_i(v)} \]

is nondecreasing in \( v \)

Conclude

When distributions are regular, the VV maximizing auction (aka Myerson’s optimal auction) is the revenue-optimal BIC mechanism!
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Conclude

When distributions are regular, the VV maximizing auction (aka Myerson’s optimal auction) is the revenue-optimal BIC mechanism!

- Most natural dists are regular (Gaussian, uniform, exp, etc).
- Can be extended to non-regular distributions via ironing, which we will not discuss now (if at all).
Myerson’s optimal auction is noteworthy for many reasons

- Matches practical experience: when players i.i.d regular, optimal auction is Vickrey with reserve price $\phi^{-1}(0)$.
- Applies to single parameter problems more generally
- Revenue maximization reduces to welfare maximization for these problems
- The optimal BIC mechanism just so happens to be DSIC and deterministic!!