CS675: Convex and Combinatorial Optimization
Spring 2018
The Simplex Algorithm

Instructor: Shaddin Dughmi
We will look at 2 algorithms in detail: Simplex and Ellipsoid.

If there is time, we might also look at interior point methods (e.g. gradient descent and variants). These are important in practice.
First methodical procedure for solving linear programs
Developed by George Dantzig in 1947
Considered one of the most influential algorithms of the 20th century
History and Basics of the Simplex Algorithm

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- Really a family of algorithms, parametrized by a “pivot rule”
- Efficient in practice, leading to conjectures that it runs in polynomial time
- In 1972, Klee and Minty exhibited worst-case examples that take exponential time, at least for some of the most popular simplex pivot rules
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In 1972, Klee and Minty exhibited worst-case examples that take exponential time, at least for some of the most popular simplex pivot rules
This spurred development of the Ellipsoid method, interior point methods, . . .
Outline

1. Description of The Simplex Algorithm
2. Properties
3. Initialization
We consider a standard form LP written as follows for convenience:

\[
\begin{align*}
\text{maximize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b
\end{align*}
\]

We use \( n \) to denote the number of variables, and \( m \) to denote the number of constraints.

Recall: optimal occurs at a vertex and corresponds to \( n \) linearly-independent tight inequalities. We assume we are given a starting vertex \( x_0 \) as input, and want to compute optimal vertex \( x^* \) This is Phase II. Phase I, finding an initial vertex, involves solving another LP. We will come back to this at the end.

Degeneracy: a vertex with >\( n \) tight inequalities. We will mostly assume this away to save ourselves a headache. Incidentally, algorithm will produce optimal dual \( y^* \) as well.
Linear Programming

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\begin{align*}
\text{maximize} & \quad c^\top x \\
\text{subject to} & \quad Ax \preceq b \\
\text{minimize} & \quad y^\top b \\
\text{subject to} & \quad y^\top A = c^\top \\
& \quad y \succeq 0
\end{align*}
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- Incidentally, algorithm will produce optimal dual \( y^* \) as well.
Apply force field $c$ to a ball inside bounded polytope $Ax \leq b$. 

Eventually, the ball will come to rest against the walls of the polytope.

Wall $a_i x \leq b_i$ applies some force $-y_i a_i$ to the ball for some $y_i \geq 0$.

Since the ball is still, $c^T = \sum_i y_i a_i = y^T A$.

At optimality, only the walls adjacent to the ball push (Complementary Slackness). Necessary and sufficient for optimality, given dual-feasible $y$. 

Description of The Simplex Algorithm 4/12
Recall: Physical Interpretation of LP

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- At optimality, only the walls adjacent to the ball push (Complementary Slackness)
  - Necessary and sufficient for optimality, given dual-feasible $y$.
Informal Description

- Starts at initial vertex \( x = x_0 \)
- While \( x \) is not optimal, move to a neighbouring vertex \( x' \) with \( cx' > cx \).
Informal Description

- Starts at initial vertex $x = x_0$
- While $x$ is not optimal, move to a neighbouring vertex $x'$ with $cx' > cx$.
  - Either $c$ is in the cone defined by tight constraints at $x$, in which case $x$ is optimal by complementary slackness
  - Or else can improve $cx$ by moving along an edge (1-d face)
Simplex Method

- **Input:** vertex $x = x_0$
- **Output:** Optimal vertex $x^*$ and complementary dual $y^*$, or unbounded

**Repeat the following:**

1. Write $c^T = y^T A$, where $y_i \neq 0$ only for $n$ tight constraints $a_i x = b_i$.
2. If $y \geq 0$ then **stop and return** $(x, y)$, else
3. Choose $i$ with $y_i < 0$, and let $d$ be s.t. $A_T \{i\} d = 0$ and $a_i d = -1$.
4. If $x + \lambda d$ feasible for all $\lambda \geq 0$, **stop and return unbounded**, else
5. $x \leftarrow x + \lambda d$, for largest $\lambda \geq 0$ maintaining feasibility
Simplex Method

- **Input:** vertex $x = x_0$
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Repeat the following:

1. Write $c^\top = y^\top A$, where $y_i \neq 0$ only for $n$ tight constraints $a_i x = b_i$.
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- Let $T$ be set of tight rows. $y_T^\top A_T = c^\top$
- Gaussian elimination
Simplex Method

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- $y$ is a dual satisfying complementary slackness with $x$
- Therefore both are optimal
**Simplex Method**

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2. If $y \geq 0$ then **stop and return** $(x, y)$, else
3. Choose $i$ with $y_i < 0$, and let $\vec{d}$ be s.t. $AT\{i\}d = 0$ and $a_id = -1$.
4. If $x + \lambda d$ feasible for all $\lambda \geq 0$, **stop and return unbounded**, else
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- Chosen so that moving in direction $d$ preserves tightness of $T \setminus \{i\}$, and loosens $i$.
- $A_T$ is full-rank, therefore $null(A_T\{i\})$ is a 1-dimensional subspace which is not normal to $a_i$
- Choose $d \in null(A_T\{i\})$ appropriately.
- Moving in direction $d$ improves objective: $c^\top d = y^\top Ad = y_ia_id > 0$
Simplex Method

**Input:** vertex $x = x_0$

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- i.e. $Ad \leq 0$
Simplex Method

- **Input:** vertex \( x = x_0 \)
- **Output:** Optimal vertex \( x^* \) and complementary dual \( y^* \), or unbounded

Repeat the following:

1. Write \( c^\top = y^\top A \), where \( y_i \neq 0 \) only for \( n \) tight constraints \( a_i x = b_i \).
2. If \( y \geq 0 \) then **stop and return** \((x, y)\), else
3. Choose \( i \) with \( y_i < 0 \), and let \( d \) be s.t. \( A_{T \setminus \{ i \}} d = 0 \) and \( a_i d = -1 \).
4. If \( x + \lambda d \) feasible for all \( \lambda \geq 0 \), **stop and return unbounded**, else
5. \( x \leftarrow x + \lambda d \), for largest \( \lambda \geq 0 \) maintaining feasibility

- \( \lambda = \min \left\{ \frac{b_j - a_j x}{a_j d} : j \in [m], a_j d > 0 \right\} \)
- \( j \) achieving this minimum is a new tight constraint, replacing \( i \).
- By nondegeneracy assumption, \( \lambda > 0 \)
Outline

1. Description of The Simplex Algorithm
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Correctness

Claim

If the simplex algorithm terminates, then it correctly outputs either an optimal primal/dual pair or unbounded.

- Primal feasibility of $x$ is maintained throughout
- Returns $(x, y)$ only if $y$ is dual feasible and satisfies complementary slackness
  - $x$ and $y$ are both optimal
- Returns unbounded only if there is a direction $d$ with $c^T d > 0$ and $A d \leq 0$. 

Termination in the Absence of Degeneracy

Claim
In the absence of degenerate vertices, the simplex algorithm terminates in a finite number of steps, at most \( \binom{m}{n} \leq 2^m \).

- There are at most \( \binom{m}{n} \) distinct vertices in the polyhedron.
- In the absence of degeneracy, the simplex algorithm does not repeat a vertex.
  - In each iteration, moves along an edge in direction \( d \), in total \( \lambda d \).
  - We saw: \( c^\top d > 0 \), and \( \lambda > 0 \).
  - Objective strictly improves each iteration.
The algorithm we presented was not fully specified.

- When multiple neighboring vertices are improving, which one should we choose so as to terminate as quickly as possible?
- In the presence of degeneracy, how should we identify the next (geometric) vertex so as to guarantee termination?
  - We maintain \( n \) tight and linearly independent constraints \( T \), to be thought of as an algebraic representation of a vertex (aka a basic feasible solution (BFS)).
  - When many algebraic representations are possible of a single geometric vertex, unclear how to identify the next geometric vertex.
Both concerns are addressed by the use of a pivot rule, which determines the order in which we examine algebraic vertices.

**Pivot rule**

A rule for selecting which $i$ leaves $T$, and which $j$ enters $T$, when multiple choices are possible either because of multiple improving neighbors or degeneracy. Examples:

- Bland’s rule: Choose lowest indexed $i$, then lowest indexed $j$
- Lexicographic: Maintain an order over rows, and move from $T$ to the lexicographically smallest possible $T'$.
- Perturbation: perturb entries of $b$ by a small value to remove degeneracy. This perturbation can be purely symbolic.
Many pivot rules, like the ones we mentioned, have been shown to never cycle over algebraic vertices

- Guarantees termination in general, even in the presence of degeneracies
- See book and notes for proofs.

However, no pivot rules have been shown to guarantee a polynomial number of pivots

- Even if no degeneracies.

In 1972, Klee and Minty exhibited a family of examples that lead to exponential worst-case runtime for some widely-used pivot rules
Nevertheless, one explanation as to the efficiency of the simplex algorithm in practice is through **smoothed complexity**

### Theorem (Spielman & Teng ’01)

*The simplex algorithm has polynomial smoothed complexity.*

- **Model of input:**
  - $A, b, c$ chosen arbitrarily (worst case)
  - Then subjected to small gaussian noise with stddev $\sigma$ (relative to largest entry of $A, b, c$)
  - Interpretation: measurement error

- More optimistic than worst case, but not quite as optimistic as average case.
- Expected runtime is polynomial in $n, m$ and $\frac{1}{\sigma}$
Open Question

Is there a pivot rule which guarantees a polynomial number of pivots of the simplex algorithm in the worst case?

Why is this important?

- Would yield a **strongly** polynomial algorithm for LP
- If true, resolves in the affirmative a classic open question in polyhedral combinatorics
  - **Polynomial Hirsch Conjecture**: Is the diameter of the edge-vertex graph of an $m$-facet polytope in $n$-dimensional space bounded by a polynomial in $n$ and $m$?
Outline

1. Description of The Simplex Algorithm
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Solving a Linear Program via the Simplex Method

- Phase I: Find a vertex $x_0$.
- Phase II: Run the simplex algorithm starting from $x_0$.

So far, we have looked only at phase II.

For phase I, we pose a different LP whose optimal solution is a vertex, if one exists.
Phase I

maximize \( c^\top x \)
subject to
\[ Ax \leq b \]
\[ x \geq 0 \]

If \( x = 0 \) is feasible, then it is a vertex and we are done, otherwise \( b_{\text{min}} < 0 \).
Phase I

maximize \( c^\top x \) subject to \( Ax \leq b \) \( x \geq 0 \)

minimize \( z \) subject to \( Ax - z \vec{1} \leq b \) \( x \geq 0 \) \( z \geq 0 \)

- If \( x = 0 \) is feasible, then it is a vertex and we are done, otherwise \( b_{\text{min}} < 0 \)
- We write a new LP with a variable \( z \) measuring how far we are from feasibility
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- We write a new LP with a variable $z$ measuring how far we are from feasibility
- If original LP is feasible, then an optimal solution of the new LP will have $z = 0$ and yield a feasible solution for original LP.
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- We write a new LP with a variable $z$ measuring how far we are from feasibility
- If original LP is feasible, then an optimal solution of the new LP will have $z = 0$ and yield a feasible solution for original LP.
- An optimal vertex of new LP (with $z = 0$) will correspond to some vertex $x_0$ of original LP
Phase I

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$x \geq 0$
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- We need a starting vertex for new LP, this is easier!

- Let $x_0' = 0$, and $z_0 = -b_{\text{min}}$
Phase I

maximize \( c^\top x \)
subject to \( Ax \preceq b \)
\( x \succeq 0 \)

minimize \( z \)
subject to \( Ax - z\mathbf{1} \preceq b \)
\( x \succeq 0 \)
\( z \geq 0 \)

- We need a starting vertex for new LP, this is easier!
  - Let \( x'_0 = 0 \), and \( z_0 = -b_{\min} \)
  - Running simplex on new LP with starting vertex \((x'_0, z_0)\), we get starting vertex \( x_0 \) for original LP.