General Instructions  The following assignment is meant to be challenging. Feel free to discuss with fellow students, though please write up your solutions independently and acknowledge everyone you discussed the homework with on your writeup. I also expect that you will not attempt to consult outside sources, on the Internet or otherwise, for solutions to any of these homework problems.

Several of these problems are drawn from Boyd and Vendenberghe. I have divided the problems into three sets: easy, medium, and difficult. Finally, whenever a question asks you to “show” or “prove” a claim, please provide a formal mathematical proof.

1 Easy Problems

Problem 1. (2 points)
B&V Exercise 3.2.

Problem 2. (4 points)
B&V Exercise 3.10.

Problem 3. (4 points)
B&V Exercise 3.21.

Problem 4. (4 points)
B&V Exercise 3.29.

Problem 5. (4 points)
B&V Exercise 4.15.

Problem 6. (4 points)
B&V Exercise 4.30.

Problem 7. (3 points)
B&V Exercise 4.59.
Problem 8. (4 points)
B&V Exercise 5.1.

Problem 9. (4 points)
B&V Exercise 5.21.

Problem 10. (3 points)
B&V Exercise 5.22.

2 Medium Problems

Problem 11. (4 points)

Problem 12. (4 points)
B&V Exercise 3.30.

Problem 13. (4 points)
B&V Exercise 3.36, parts a and c.

Problem 14. (5 points)
B&V Exercise 4.25.

Problem 15. (6 points)
B&V Exercise 4.43.

Problem 16. (4 points)
B&V Exercise 4.56. (Hint: perspective of a convex function is convex)

Problem 17. (4 points)
B&V Exercise 5.13.

Problem 18. (4 points)
B&V Exercise 5.18.

Problem 19. (4 points)
B&V Exercise 5.30.

Problem 20. (6 points)
In this problem, we will prove some facts about geometric duality.

(a) Show that the polar of the $p$-norm ball is the $q$-norm ball, when $\frac{1}{p} + \frac{1}{q} = 1.$
(b) Characterize sets $\mathcal{P} \subseteq \mathbb{R}^n$ which are self polar, i.e. which satisfy $\mathcal{P} = \mathcal{P}^\circ$. Prove that your characterization is correct.

(c) Describe the polar of an ellipsoid centered at the origin.

3 Difficult Problems

Problem 21. (5 points)
B&V Exercise 5.37

Problem 22. (6 points)
B&V Exercise 5.39

Problem 23. (6 points)
Let $f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$ be convex functions. Let $g : \mathbb{R}^n \to \mathbb{R}$ be the greatest convex function lower-bounding their point-wise minimum, i.e. $g(x) = \text{convexhull}(\min_{i=1}^{m} f_i(x))$. Formulate the problem of evaluating $g(x)$ for a given $x$ as a convex program. The number of variables in your convex program should be polynomially bounded in $n$ and $m$.

Problem 24. (14 points)
In this problem, we will explore the relationship between polar and Lagrangian duality. Specifically, in the context of linear programming, we will argue that the two are respectively geometric and algebraic formulations of the same idea.

a (2 points). Consider a linear program of the form

$$\begin{align*}
\text{maximize} & \quad c^T x \\
\text{subject to} & \quad Ax \preceq b
\end{align*} \tag{1}$$

Using the rules we saw in class, we can derive the Lagrangian dual of (1) as the following LP

$$\begin{align*}
\text{minimize} & \quad b^T y \\
\text{subject to} & \quad A^T y = c \\
& \quad y \succeq 0
\end{align*} \tag{2}$$

Naturally, scaling each inequality $a_i \cdot x \leq b_i$ of (1) by a constant $\alpha_i > 0$ to get the inequality $\alpha_i a_i \cdot x \leq \alpha_i b_i$ preserves the feasible set and objective function of (1) (and therefore also preserves the optimal solution and objective value). In other words, scaling the inequalities produces a geometrically equivalent optimization problem. Show that the same cannot be said for the Lagrangian dual of (1); specifically, show that scaling the inequalities of (1) changes feasible set and optimal solution of its dual. Conclude that Lagrangian duality is an algebraic transformation, since given two equivalent LPs (same feasible set and objective) represented differently, it yields different dual LPs.

b (2 points). For simplicity, assume $x = \bar{x}$ is a strictly feasible solution of (1) — i.e., the feasible
region includes an open ball about the origin. Show that (1) is equivalent, in the sense of having the same feasible set and objective function, to an LP of the following “normalized” form, and has an optimal value $\nu^* \geq 0$.

$$\begin{align*}
\text{maximize} & \quad c^T x \\
\text{subject to} & \quad \hat{A} x \preceq \vec{1}
\end{align*}$$

(3)

c (2 points). Using the rules for taking duals, the dual of (3) is the following LP.

$$\begin{align*}
\text{minimize} & \quad \vec{1}^T y \\
\text{subject to} & \quad \hat{A}^T y = c \\
& \quad y \succeq \vec{0}
\end{align*}$$

(4)

Show that (4) and (2) are equivalent up to a simple transformation of the variables, and note that said transformation preserves the optimal value.

d (4 points). Let $P = \{ x \in \mathbb{R}^n : \hat{A} x \preceq \vec{1} \}$ denote the feasible set of (3) (and therefore also of (1)), and let $P^\circ$ denote its polar. You will show that one can derive tight bounds on $\nu^*$ from the polar $P^\circ$. Specifically, show that if $\frac{1}{\nu} c \in P^\circ$ for some constant $\nu > 0$, then $\nu^* \leq \nu$. Conversely, show that $\frac{1}{\nu^*} c \in P^\circ$.

e (4 points). Recall from class that if $P = \{ x \in \mathbb{R}^n : \hat{A} x \preceq \vec{1} \}$ is a polytope, then its polar $P^\circ$ is equal to the convex hull of the rows of the matrix $\hat{A}$. More generally, when $P$ is a polyhedron its polar $P^\circ$ is the convex hull of $\text{rows}(\hat{A}) \cup \{\vec{0}\}$ (you are invited to verify this for yourself if curious). Explain how LP (4) — and by (c), also LP (2) — can be interpreted as finding the tightest upperbound on $\nu^*$ implied by the polar, in the sense of (d). Conclude that Lagrangian duality is an algebraic analog of polar duality, which is a purely geometric relationship between convex sets.

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1This is without loss of generality in most (though not all) natural applications of LP, since it can be enforced by a combination of projection and a suitable shift of the feasible set.