General Instructions  The following assignment is meant to be challenging. Feel free to discuss with fellow students, though please write up your solutions independently and acknowledge everyone you discussed the homework with on your writeup. I also expect that you will not attempt to consult outside sources, on the Internet or otherwise, for solutions to any of these homework problems — doing so would be considered cheating.

Several of these problems are drawn from the following texts, each of which is linked on the course website: Luenberger and Ye (4th edition), Korte and Vygen (5th edition), and Boyd and Vendenberghe. Please make sure you are using the correct edition of each of the books by using the links on the course website. I have divided the problems into three sets: short problems, which are intended to take you 5-15 minutes each, medium problems, intended to take 15-30 minutes each, and long problems, which may take anywhere between 30 minutes to several hours depending on whether inspiration strikes.

Finally, whenever a question asks you to “show” or “prove” a claim, please provide a formal mathematical proof.

1  Short Problems \( (5 - 15 \text{ minutes each}) \)

Problem 1.  (3 points)
L&Y Chapter 2, Exercise 2.

Problem 2.  (3 points)
L&Y Chapter 2, Exercise 8.

Problem 3.  (3 points)
L&Y Chapter 2, Exercise 9.

Problem 4.  (4 points)
L&Y Chapter 2, Exercise 10.

Problem 5.  (3 points)
K&V Chapter 3, Exercise 4.
Problem 6. (3 points)
K&V Chapter 3, Exercise 15. Though the theorem is true regardless, for simplicity you may assume that $X$ is a finite set.
(Hint: Write an LP expressing $y$ as a convex combination of points in $X$, then use our trick for counting the number of non-zero variables).

Problem 7. (3 points)
B&V Exercise 2.3.

Problem 8. (3 points)
B&V Exercise 2.21.
2 Medium Problems (15 – 30 minutes each)

Problem 9. (5 points)
The strict separating hyperplane theorem states that whenever \( A, B \subseteq \mathbb{R}^n \) are disjoint closed convex sets, and at least one of them is compact, there is a hyperplane strictly separating them; i.e. there is a vector \( a \in \mathbb{R}^n \) and \( b \in \mathbb{R} \) such that \( a^T x < b \) for every \( x \in A \), and \( a^T x > b \) for every \( x \in B \). The Farkas Lemma states that for every matrix \( A \in \mathbb{R}^{n \times m} \) and vector \( b \), either the system \( Ax = b \) has a nonnegative vector solution \( x \succeq 0 \), or else there is a vector \( y \) such that \( y^T A \succeq 0 \) and \( y^T b < 0 \). Show that the Farkas lemma follows from the strict separating hyperplane theorem.

Problem 10. (5 points)
B&V Exercise 2.12.

Problem 11. (5 points)
B&V Exercise 2.15.

Problem 12. (5 points)
B&V Exercise 2.16.

Problem 13. (5 points)

Problem 14. (5 points)
B&V Exercise 2.36. Additionally, describe the dual cone of Euclidean distance matrices as the conic hull of a set of matrices.
(Note: The latter part of the question will make more sense after we cover geometric duality in class.)

Problem 15. (5 points)
Let \( P \subseteq \mathbb{R}^n \) be a polytope, and let \( A \) be a full-rank \( n \times n \) matrix. Let \( P' = \{ Ax : x \in P \} \) be the image of \( P \) under \( A \). Show that \( x \) is a vertex of \( P \) if and only if \( Ax \) is a vertex of \( P' \).
3 Long Problems (> 30 minutes each)

Problem 16. (10 points)
L&Y Chapter 4, Exercise 9. In part (b), additionally show that Y is guaranteed payoff at least $-B$ no matter what $x$ is chosen by $X$.

Clarification: Note that each $a_{ij}$ may be positive or negative, where a negative entries means that player $Y$ wins amount $-a_{ij}$ from $X$. The average payoff to player $Y$ is the negation of the average payoff to player $X$, namely $-x^T A y$.

Problem 17. (8 points)
Given a matrix $A \in \mathbb{R}^{m \times n}$, show that exactly one of the following systems has a solution

- $Ax > 0, x \in \mathbb{R}^n$ (note: $Ax$ is entry-wise strictly greater than zero)
- $A^T y = 0, y \succeq 0$, and $y \in \mathbb{R}^m$ is non-zero

Problem 18. (8 points)
Recall the optimal production problem we introduced in class. We will now consider a generalization of that problem to a setting with multiple colluding firms. As in the optimal production problem, there are $n$ products and $m$ resources, where producing a unit of the $j$'th product consumes $A_{ij}$ units of the $i$'th resource, and each unit of the $j$'th product can be sold at a profit of $c_j$. There are $K$ firms, the $k$'th of whom is endowed with $b_k^i$ units of the $i$'th resource — we use $b_k \in \mathbb{R}_+^m$ to denote the endowment of the $k$'th firm. We allow a subset $S \subseteq \{1, \ldots, K\}$ of the firms to collude, in which case the firms pool their resources in order to maximize their collective profit, effectively solving the following optimization problem.

$$OPT(S) = \maximize \quad e^T x$$
$$\text{subject to} \quad Ax \preceq \sum_{k \in S} b_k^i, \quad x \succeq 0$$

We say a coalition of firms is stable if the firms can share profit in such a way so that no subset of the coalition can gain by breaking with the group and forming a coalition of their own. Formally, coalition $S \subseteq \{1, \ldots, K\}$ is stable if there are profit shares $p \in \mathbb{R}_+^S$ such that

1. $\sum_{k \in S} p_k = OPT(S)$. i.e. the total profit distributed equals the aggregate profit of the coalition.

2. $\sum_{k \in T} p_k \geq OPT(T)$ for all $T \subseteq S$. i.e. no subset of the coalition can collectively increase their profit by breaking off from $S$.

Show that the grand coalition $\{1, \ldots, K\}$ is stable.