Convex Sets

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Outline

1. Convex sets, Affine sets, and Cones
2. Examples of Convex Sets
3. Convexity-Preserving Operations
4. Separation Theorems
A set $S \subseteq \mathbb{R}^n$ is **convex** if the line segment between any two points in $S$ lies in $S$. i.e. if $x, y \in S$ and $\theta \in [0, 1]$, then $\theta x + (1 - \theta)y \in S$. 

**Convex Sets**
Convex Sets

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Equivalent Definition

$S$ is convex if every convex combination of points in $S$ lies in $S$.

Convex Combination

- Finite: $y$ is a convex combination of $x_1, \ldots, x_k$ if
  
  $y = \theta_1 x_1 + \ldots + \theta_k x_k$, where $\theta_i \geq 0$ and $\sum_i \theta_i = 1$.

- General: expectation of probability measure on $S$.
Convex Sets

Convex Hull

The convex hull of $S \subseteq \mathbb{R}^n$ is the smallest convex set containing $S$.

- Intersection of all convex sets containing $S$
- The set of all convex combinations of points in $S$
Convex Sets

**Convex Hull**

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A set $S$ is convex if and only if $\text{convexhull}(S) = S$. 
A set $S \subseteq \mathbb{R}^n$ is affine if the line passing through any two points in $S$ lies in $S$. i.e. if $x, y \in S$ and $\theta \in \mathbb{R}$, then $\theta x + (1 - \theta)y \in S$.

Obviously, affine sets are convex.
Affine Set

A set \( S \subseteq \mathbb{R}^n \) is **affine** if the line passing through any two points in \( S \) lies in \( S \). i.e. if \( x, y \in S \) and \( \theta \in \mathbb{R} \), then \( \theta x + (1 - \theta)y \in S \).

![Diagram showing affine combination](image)

Obviously, affine sets are convex.

Equivalent Definition

\( S \) is affine if every **affine combination** of points in \( S \) lies in \( S \).

Affine Combination

\( y \) is an affine combination of \( x_1, \ldots, x_k \) if \( y = \theta_1 x_1 + \ldots + \theta_k x_k \), and \( \sum_i \theta_i = 1 \).

Generalizes convex combinations
Affine Sets

Equivalent Definition II

$S$ is affine if and only if it is a shifted subspace

- i.e. $S = x_0 + V$, where $V$ is a linear subspace of $\mathbb{R}^n$.

- Any $x_0 \in S$ will do, and yields the same $V$.
- The dimension of $S$ is the dimension of subspace $V$. 

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Affine Sets

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Equivalent Definition III

$S$ is affine if and only if it is the solution of a set of linear equations (i.e. the intersection of hyperplanes).

- i.e. $S = \{x : Ax = b\}$ for some matrix $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. 
Affine Sets

Affine Hull

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**Affine Sets**

**Affine Hull**

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A set $S$ is affine if and only if $\text{affinehull}(S) = S$. 
Affine Sets

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Affine Dimension

The **affine dimension** of a set is the dimension of its affine hull
A set $K \subseteq \mathbb{R}^n$ is a cone if the ray from the origin through every point in $K$ is in $K$ i.e. if $x \in K$ and $\theta \geq 0$, then $\theta x \in K$.

Note: every cone contains 0.
Cones

A set $K \subseteq \mathbb{R}^n$ is a cone if the ray from the origin through every point in $K$ is in $K$ i.e. if $x \in K$ and $\theta \geq 0$, then $\theta x \in K$.

Note: every cone contains 0.

Special Cones

- A convex cone is a cone that is convex
- A cone is pointed if whenever $x \in K$ and $x \neq 0$, then $-x \not\in K$.
- We will mostly mention proper cones: convex, pointed, closed, and of full affine dimension.
Equivalent Definition

$K$ is a convex cone if every conic combination of points in $K$ lies in $K$.

Conic Combination

$y$ is a conic combination of $x_1, \ldots, x_k$ if $y = \theta_1 x_1 + \ldots + \theta_k x_k$, where $\theta_i \geq 0$. 
Cones

**Conic Hull**

The conic hull of $K \subseteq \mathbb{R}^n$ is the smallest convex cone containing $K$

- Intersection of all convex cones containing $K$
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Cones

Conic Hull

The conic hull of \( K \subseteq \mathbb{R}^n \) is the smallest convex cone containing \( K \)
- Intersection of all convex cones containing \( K \)
- The set of all conic combinations of points in \( K \)

A set \( K \) is a convex cone if and only if \( \text{conichull}(K) = K \).
A cone is polyhedral if it is the set of solutions to a finite set of homogeneous linear inequalities $Ax \leq 0$. 

Convex sets, Affine sets, and Cones
- Linear Subspace: Affine, Cone
- Hyperplane: Affine, cone if includes 0
- Halfspace: Cone if origin on boundary
- Line: Affine, cone if includes 0
- Ray: Cone if endpoint at 0
- Line segment
- Polyhedron: finite intersection of halfspaces

- Polytope: Bounded polyhedron
• Nonnegative Orthant $\mathbb{R}^n_+$: Polyhedral cone
• Simplex: convex hull of affinely independent points
  • Unit simplex: $x \succeq 0, \sum_i x_i \leq 1$
  • Probability simplex: $x \succeq 0, \sum_i x_i = 1$. 
Euclidean ball: \( \{ x : \| x - x_c \|_2 \leq r \} \) for center \( x_c \) and radius \( r \)

Ellipsoid: \( \{ x : (x - x_c)^T P^{-1} (x - x_c) \leq 1 \} \) for symmetric \( P \succeq 0 \)
- Equivalently: \( \{ x_c + Au : \| u \|_2 \leq 1 \} \) for some linear map \( A \)
- **Norm ball:** \( \{ x : \| x - c \| \leq r \} \) for any norm \( \| . \| \)

The unit sphere for different metrics: \( \| x \|_{l_p} = 1 \) in \( \mathbb{R}^2 \).
● Norm ball: \( \{ x : \| x - c \| \leq r \} \) for any norm \( \| \cdot \| \)

The unit sphere for different metrics: \( \| x \|_{l_p} = 1 \) in \( \mathbb{R}^2 \).

● Norm cone: \( \{(x, r) : \| x \| \leq r\} \)

● Cone of symmetric positive semi-definite matrices \( M \)
  - Symmetric matrix \( A \succeq 0 \) iff \( x^T A x \geq 0 \) for all \( x \)
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The intersection of two convex sets is convex. This holds for the intersection of an infinite number of sets.

Examples
- Polyhedron: intersection of halfspaces
- PSD cone: intersection of linear inequalities $z^T A z \geq 0$, for all $z \in \mathbb{R}^n$. 
The intersection of two convex sets is convex. This holds for the intersection of an infinite number of sets.

Examples

- Polyhedron: intersection of halfspaces
- PSD cone: intersection of linear inequalities $z^T A z \geq 0$, for all $z \in \mathbb{R}^n$.

In fact, we will see that every closed convex set is the intersection of a (possibly infinite) set of halfspaces.
Affine Maps

If \( f : \mathbb{R}^n \to \mathbb{R}^m \) is an affine function (i.e. \( f(x) = Ax + b \)), then

- \( f(S) \) is convex whenever \( S \subseteq \mathbb{R}^n \) is convex
- \( f^{-1}(T) \) is convex whenever \( T \subseteq \mathbb{R}^m \) is convex

\[
\begin{align*}
f(\theta x + (1 - \theta)y) &= A(\theta x + (1 - \theta)y) + b \\
&= \theta(Ax + b) + (1 - \theta)(Ay + b)) \\
&= \theta f(x) + (1 - \theta)f(y)
\end{align*}
\]
Examples

- An ellipsoid is image of a unit ball after an affine map.
- A polyhedron $Ax \leq b$ is inverse image of nonnegative orthant under $f(x) = b - Ax$. 
Perspective Function

Let $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be $P(x, t) = x/t$.

- $P(S)$ is convex whenever $S \subseteq \mathbb{R}^{n+1}$ is convex
- $P^{-1}(T)$ is convex whenever $T \subseteq \mathbb{R}^n$ is convex
Perspective Function

Let $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be $P(x, t) = \frac{x}{t}$.

- $P(S)$ is convex whenever $S \subseteq \mathbb{R}^{n+1}$ is convex
- $P^{-1}(T)$ is convex whenever $T \subseteq \mathbb{R}^n$ is convex

Generalizes to linear fractional functions $f(x) = \frac{Ax + b}{c^T x + d}$

- Composition of perspective with affine.
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Separating Hyperplane Theorem

If $A, B \subseteq \mathbb{R}^n$ are disjoint convex sets, then there is a hyperplane weakly separating them. That is, there is $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $a^T x \leq b$ for every $x \in A$ and $a^T y \geq b$ for every $y \in B$. 
Separating Hyperplane Theorem (Strict Version)

If $A, B \subseteq \mathbb{R}^n$ are disjoint closed convex sets, and at least one of them is compact, then there is a hyperplane strictly separating them. That is, there is $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $a^\top x < b$ for every $x \in A$ and $a^\top y > b$ for every $y \in B$. 
Farkas’ Lemma

Let $K$ be a closed convex cone and let $w \notin K$. There is $z \in \mathbb{R}^n$ such that $z^T x \geq 0$ for all $x \in K$, and $z^T w < 0$. 
Supporting Hyperplane Theorem.

If $S \subseteq \mathbb{R}^n$ is a closed convex set and $y$ is on the boundary of $S$, then there is a hyperplane supporting $S$ at $y$. That is, there is $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $a^T x \leq b$ for every $x \in S$ and $a^T y = b$. 

Separation Theorems