Combinatorial Problems as Linear and Convex Programs

Instructor: Shaddin Dughmi
Outline

1. Introduction
2. Shortest Path
4. Bipartite Matching
5. Total Unimodularity
6. Duality of Bipartite Matching and its Consequences
7. Spanning Trees
8. Flows
9. Max Cut
In CS, discrete problems are traditionally viewed/analyzed using discrete mathematics and combinatorics.

- Algorithms are combinatorial in nature (greedy, dynamic programming, divide and conquer, etc).

Increasingly in recent history, it is becoming clear that combining both viewpoints is the way to go.

- Better algorithms (runtime, approximation)
- Structural insights (e.g. market clearing prices in matching markets)
- Unifying theories and general results (Matroids, submodular optimization, constraint satisfaction)
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This is not surprising, since almost any finite family of discrete objects can be encoded as a finite subset of Euclidean space. Convex hull of that set is a polytope. E.g. spanning trees, paths, cuts, TSP tours, assignments...
LP algorithms typically require representation as a “small” family of inequalities,
- Not possible in general (Say when problem is NP-hard, assuming \((P \neq NP)\))
- Shown unconditionally impossible in some cases (e.g. TSP)
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But, in many cases, polyhedra in inequality form can be shown to encode a combinatorial problems at the vertices

Next

We examine some combinatorial problems through the lens of LP and convex optimization, starting with shortest path.
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The Shortest Path Problem

Given a directed graph $G = (V, E)$ with cost $c_e \in \mathbb{R}$ on edge $e$, find the minimum cost path from $s$ to $t$.

- We use $n$ and $m$ to denote $|V|$ and $|E|$, respectively.
- We allow costs to be negative, but assume no negative cycles.
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- We allow costs to be negative, but assume no negative cycles.

When costs are nonnegative, Dijkstra’s algorithm finds the shortest path from \( s \) to every other node in time \( O(m + n \log n) \).

Using primal/dual paradigm, we will design a polynomial-time algorithm that works when graph has negative edges but no negative cycles.
When the graph has no negative cycles, there is a shortest path which is simple.

When the graph has negative cycles, there may not be a shortest path from $s$ to $t$.

In these cases, the algorithm we design can be modified to “fail gracefully” by detecting such a cycle.

- Can be used to detect arbitrage opportunities in currency exchange networks.
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In the presence of negative cycles, finding the shortest simple path is NP-hard (by reduction from Hamiltonian cycle).
Consider the following LP

### Primal Shortest Path LP

\[
\begin{align*}
\min \quad & \sum_{e \in E} c_e x_e \\
\text{s.t.} \quad & \sum_{e \rightarrow v} x_e - \sum_{v \rightarrow e} x_e = \delta_v, \quad \forall v \in V. \\
& x_e \geq 0, \quad \forall e \in E.
\end{align*}
\]

where \( \delta_v = -1 \) if \( v = s \), 1 if \( v = t \), and 0 otherwise.
Consider the following LP

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- This is a relaxation of the shortest path problem
  - Indicator vector \(x_P\) of \(s - t\) path \(P\) is a feasible solution, with cost as given by the objective
  - Fractional feasible solutions may not correspond to paths
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- This is a relaxation of the shortest path problem
  - Indicator vector \(x_P\) of \(s - t\) path \(P\) is a feasible solution, with cost as given by the objective
  - Fractional feasible solutions may not correspond to paths
- A-priori, it is conceivable that optimal value of LP is less than length of shortest path.
Integrality of the Shortest Path Polyhedron

\[
\begin{align*}
\text{min} & \quad \sum_{e \in E} c_e x_e \\
\text{s.t.} & \quad \sum_{e \rightarrow v} x_e - \sum_{v \rightarrow e} x_e = \delta_v, \quad \forall v \in V. \\
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We will show that above LP encodes the shortest path problem exactly.

**Claim**

When $c$ satisfies the no-negative-cycles property, the indicator vector of the shortest $s - t$ path is an optimal solution to the LP.
We will use the following LP dual

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### Dual LP

\[
\begin{align*}
\text{max} & \quad y_t - y_s \\
\text{s.t.} & \quad y_v - y_u \leq c_e, \quad \forall (u, v) \in E.
\end{align*}
\]

- Interpretation of dual variables \( y_v \): “height” or “potential”
- Relative potential of vertices constrained by length of edge between them (triangle inequality)
- Dual is trying to maximize relative potential of \( s \) and \( t \)
Proof Using the Dual

Claim

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**Primal LP**

$$\min \sum_{e \in E} c_e x_e$$

s.t.

$$\sum_{e \rightarrow v} x_e - \sum_{v \rightarrow e} x_e = \delta_v, \quad \forall v \in V.$$  

$$x_e \geq 0, \quad \forall e \in E.$$  

**Dual LP**

$$\max y_t - y_s$$

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$$y_v - y_u \leq c_e, \quad \forall (u, v) \in E.$$
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- Let \( x^* \) be indicator vector of shortest s-t path
  - Feasible for primal
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- Let $y_v^*$ be shortest path distance from $s$ to $v$
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- Let \( x^* \) be indicator vector of shortest \( s-t \) path
  - Feasible for primal
- Let \( y^*_v \) be shortest path distance from \( s \) to \( v \)
  - Feasible for dual (by triangle inequality)
- \( \sum_e c_e x^*_e = y^*_t - y^*_s \), so both \( x^* \) and \( y^* \) optimal.
Integrality of Polyhedra

A stronger statement is true:

**Integrality of Shortest Path LP**

The vertices of the polyhedral feasible region are precisely the indicator vectors of simple paths in $G$.

- Implies that there always exists an optimal solution which is a path whenever LP is bounded and feasible
- Reduces computing shortest path in graphs with no negative cycles to finding optimal vertex of LP
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**Proof**

1. LP is bounded iff $c$ satisfies no-negative-cycles
   - $\leftarrow$: previous proof
   - $\rightarrow$: If $c$ has a negative cycle, there are arbitrarily cheap “flows” along that cycle
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Proof

1. LP is bounded iff $c$ satisfies no-negative-cycles
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2. Fact: For every LP vertex $x$ there is objective $c$ such that $x$ is unique optimal. (Prove it!)
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2. Fact: For every LP vertex $x$ there is objective $c$ such that $x$ is unique optimal. (Prove it!)

3. Since such a $c$ satisfies no-negative-cycles property, our previous claim shows that $x$ is integral.
A stronger statement is true:

Integrality of Shortest Path LP

The vertices of the polyhedral feasible region are precisely the indicator vectors of simple paths in $G$.

In general, the approach we took applies in many contexts: To show a polytope’s vertices integral, it suffices to show that there is an integral optimal for any objective which admits an optimal solution.
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Ford’s Algorithm

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\begin{align*}
\text{max } & y_t - y_s \\
\text{s.t. } & y_v - y_u \leq c_e, \quad \forall e = (u, v) \in E.
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For convenience, add \((s, v)\) of length \(\infty\) when one doesn’t exist.

**Ford’s Algorithm**

1. \(y_v = c_{(s,v)}\) and \(\text{pred}(v) \leftarrow s\) for \(v \neq s\)
2. \(y_s \leftarrow 0, \text{pred}(s) = \text{null}\).
3. While some dual constraint is violated, i.e. \(y_v > y_u + c_e\) for some \(e = (u, v)\)
   - \(y_v \leftarrow y_u + c_e\)
   - Set \(\text{pred}(v) = u\)
4. Output the path \(t, \text{pred}(t), \text{pred}(\text{pred}(t)), \ldots, s\).
Correctness

Lemma (Loop Invariant 1)

Assuming no negative cycles, \( \text{pred} \) defines a path \( P \) from \( s \) to \( t \), of length at most \( y_t - y_s \).

Interpretation

- Ford’s algorithm maintains an (initially infeasible) dual \( y \)
- Also maintains feasible primal \( P \) of length \( \leq \) dual objective \( y_t - y_s \)
- Iteratively “fixes” dual \( y \), tending towards feasibility
- Once \( y \) is feasible, weak duality implies \( P \) optimal.
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Correctness follows from loop invariant 1 and termination condition.

Theorem (Correctness)

If Ford’s algorithm terminates, then it outputs a shortest path from $s$ to $t$. 

Algorithms for Single-Source Shortest Path
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Theorem (Correctness)

If Ford’s algorithm terminates, then it outputs a shortest path from \( s \) to \( t \)

Algorithms of this form, that output a matching primal and dual solution, are called Primal-Dual Algorithms.
Termination

Lemma (Loop Invariant 2)

Assuming no negative cycles, $y_v$ is the length of some simple path from $s$ to $v$. 
Termination

Lemma (Loop Invariant 2)
Assuming no negative cycles, $y_v$ is the length of some simple path from $s$ to $v$.

Theorem (Termination)
When the graph has no negative cycles, Ford’s algorithm terminates in a finite number of steps.

Proof
- The graph has a finite number $N$ of simple paths
- By loop invariant 2, every dual variable $y_v$ is the length of some simple path.
- Dual variables are nonincreasing throughout algorithm, and one decreases each iteration.
- There can be at most $nN$ iterations.
Observation: Single sink shortest paths

Ford’s Algorithm

1. \( y_v = c_{(s,v)} \) and \( \text{pred}(v) \leftarrow s \) for \( v \neq s \)
2. \( y_s \leftarrow 0 \), \( \text{pred}(s) = \text{null} \).
3. While some dual constraint is violated, i.e. \( y_v > y_u + c_e \) for some \( e = (u, v) \)
   - \( y_v \leftarrow y_u + c_e \)
   - Set \( \text{pred}(v) = u \)
4. Output the path \( t, \text{pred}(t), \text{pred}(\text{pred}(t)), \ldots, s \).

Observation

Algorithm does not depend on \( t \) till very last step. So essentially solves the single-source shortest path problem. i.e. finds shortest paths from \( s \) to all other vertices \( v \).
We prove Loop Invariant 1 through two Lemmas

Lemma (Loop Invariant 1a)

For every node \( w \), we have
\[
    y_w - y_{\text{pred}(w)} \geq c_{\text{pred}(w),w}
\]

Proof

- Fix \( w \)
- Holds at first iteration
- Preserved by Induction on iterations
  - If neither \( y_w \) nor \( y_{\text{pred}(w)} \) updated, nothing changes.
  - If \( y_w \) (and \( \text{pred}(w) \)) updated, then \( y_w \leftarrow y_{\text{pred}(w)} + c_{\text{pred}(w),w} \)
  - \( y_{\text{pred}(w)} \) updated, it only goes down, preserving inequality.
Loop Invariant 1

Lemma (Invariant 1b)

Assuming no negative cycles, pred forms a directed tree rooted out of \( s \).

We denote this path from \( s \) to a node \( w \) by \( P(s, w) \).

Proof

- Holds at first iteration
- For a contradiction, consider iteration of first violation
  - \( v \) and \( u \) with \( y_v > y_u + c_{u,v} \)
  - \( P(s, u) \) passes through \( v \)
    - Otherwise tree property preserved by \( \text{pred}(v) \leftarrow u \)
- Let \( P(v, u) \) be the portion of \( P(s, u) \) starting at \( v \).
- By Invariant 1a, and telescoping sum, length of \( P(v, u) \) is at most \( y_u - y_v \).
- Length of cycle \( \{P(v, u), (u, v)\} \) at most \( y_u - y_v + c_{u,v} < 0 \).
Summarizing Loop Invariant 1

Lemma (Invariant 1a)
For every node $w$, we have $y_w - y_{\text{pred}(w)} \geq c_{\text{pred}(w),w}$.

- By telescoping sum, can bound $y_w - y_s$ when $\text{pred}$ leads back to $s$.

Lemma (Invariant 1b)
Assuming no negative cycles, $\text{pred}$ forms a directed tree rooted out of $s$.

- Implies that $y_s$ remains 0.

Corollary (Loop Invariant 1)
Assuming no negative cycles, $\text{pred}$ defines a path $P(s,w)$ from $s$ to each node $w$, of length at most $y_w - y_s = y_w$. 
Lemma (Loop Invariant 2)

Assuming no negative cycles, \( y_w \) is the length of some simple path \( Q(s, w) \) from \( s \) to \( w \), for all \( w \).

Proof is technical, by induction, so we will skip. Instead, we will modify Ford's algorithm to guarantee polynomial time termination.
Bellman-Ford Algorithm

The following algorithm fixes an (arbitrary) order on edges $E$

1. $y_v = c(s,v)$ and $\text{pred}(v) \leftarrow s$ for $v \neq s$
2. $y_s \leftarrow 0$, $\text{pred}(s) = \text{null}$.
3. While $y$ is infeasible for the dual
   - For $e = (u, v)$ in order, if $y_v > y_u + c_e$ then
     - $y_v \leftarrow y_u + c_e$
     - Set $\text{pred}(v) = u$
4. Output the path $t, \text{pred}(t), \text{pred}(\text{pred}(t)), \ldots, s$. 

Note: Correctness follows from the correctness of Ford’s Algorithm.
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4. Output the path $t, \text{pred}(t), \text{pred}(\text{pred}(t)), \ldots, s$.

**Note**

Correctness follows from the correctness of Ford’s Algorithm.
Bellman-Ford terminates after \( n - 1 \) scans through \( E \), for a total runtime of \( O(nm) \).
Theorem

Bellman-Ford terminates after $n - 1$ scans through $E$, for a total runtime of $O(nm)$.

Follows immediately from the following Lemma

Lemma

After $k$ scans through $E$, vertices $v$ with a shortest $s - v$ path consisting of $\leq k$ edges are correctly labeled. (i.e., $y_v = distance(s,v)$)
Proof

Lemma

After $k$ scans through $E$, vertices $v$ with a shortest $s-v$ path consisting of $\leq k$ edges are correctly labeled. (i.e., $y_v = \text{distance}(s, v)$)

Proof

- Holds for $k = 0$
- By induction on $k$.
  - Assume it holds for $k - 1$.
  - Let $v$ be a node with a shortest path $P$ from $s$ with $k$ edges.
  - $P = \{Q, e\}$, for some $e = (u, v)$ and $s-u$ path $Q$, where $Q$ is a shortest $s-u$ path and $Q$ has $k - 1$ edges.
  - By inductive hypothesis, $u$ is correctly labeled just before $e$ is scanned – i.e. $y_u = \text{distance}(s, u)$.
  - Therefore, $v$ is correctly labeled $y_v \leftarrow y_u + c_{u,v} = \text{distance}(s, v)$ after $e$ is scanned
A Note on Negative Cycles

Question
What if there are negative cycles? What does that say about LP? What about Ford’s algorithm?
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The Max-Weight Bipartite Matching Problem

Given a bipartite graph \( G = (V, E) \), with \( V = L \cup R \), and weights \( w_e \) on edges \( e \), find a maximum weight matching.

- **Matching**: a set of edges covering each node at most once
- We use \( n \) and \( m \) to denote \(|V|\) and \(|E|\), respectively.
- Equivalent to maximum weight / minimum cost perfect matching.

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- We use $n$ and $m$ to denote $|V|$ and $|E|$, respectively.
- Equivalent to maximum weight / minimum cost perfect matching.

Our focus will be less on algorithms, and more on using polyhedral interpretation to gain insights about a combinatorial problem.
Bipartite Matching LP

\[
\begin{align*}
\text{max} \quad & \sum_{e \in E} w_e x_e \\
\text{s.t.} \quad & \sum_{e \in \delta(v)} x_e \leq 1, \quad \forall v \in V. \\
& x_e \geq 0, \quad \forall e \in E.
\end{align*}
\]
An LP Relaxation of Bipartite Matching

**Bipartite Matching LP**

\[
\begin{align*}
\max & \quad \sum_{e \in E} w_e x_e \\
\text{s.t.} & \quad \sum_{e \in \delta(v)} x_e \leq 1, \quad \forall v \in V. \\
& \quad x_e \geq 0, \quad \forall e \in E.
\end{align*}
\]

- Feasible region is a polytope \( P \) (i.e. a bounded polyhedron)
- This is a **relaxation** of the bipartite matching problem
  - Integer points in \( P \) are the indicator vectors of matchings.

\[
P \cap \mathbb{Z}^m = \{ x_M : M \text{ is a matching} \} 
\]
The feasible region of the matching LP is the convex hull of indicator vectors of matchings.

\[ P = \text{convexhull} \{ x_M : M \text{ is a matching} \} \]
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\[ \mathcal{P} = \text{convexhull} \{ x_M : M \text{ is a matching} \} \]

**Note**

- This is the strongest guarantee you could hope for of an LP relaxation of a combinatorial problem
- Solving LP is equivalent to solving the combinatorial problem
- Stronger guarantee than shortest path LP from last time
Suffices to show that all vertices are integral (why?)
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Consider $x \in \mathcal{P}$ non-integral, we will show that $x$ is not a vertex.
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Let \( H \) be the subgraph formed by edges with \( x_e \in (0, 1) \)
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Let \( H \) be the subgraph formed by edges with \( x_e \in (0, 1) \). 

\( H \) either contains a cycle, or else a maximal path which is simple.
Proof

Suffices to show that all vertices are integral (why?)
Consider $x \in \mathcal{P}$ non-integral, we will show that $x$ is not a vertex.
Let $H$ be the subgraph formed by edges with $x_e \in (0, 1)$
$H$ either contains a cycle, or else a maximal path which is simple.
Proof

Case 1: Cycle $C$

- Let $C = (e_1, \ldots, e_k)$, with $k$ even
- There is $\epsilon > 0$ such that adding $\pm \epsilon (+1, -1, \ldots, +1, -1)$ to $x_C$ preserves feasibility
- $x$ is the midpoint of $x + \epsilon (+1, -1, \ldots, +1, -1)_C$ and $x - \epsilon (+1, -1, \ldots, +1, -1)_C$, so $x$ is not a vertex.
Case 2: Maximal Path $P$

- Let $P = (e_1, \ldots, e_k)$, going through vertices $v_0, v_1, \ldots, v_k$.
- By maximality, $e_1$ is the only edge of $v_0$ with non-zero $x$-weight.
  - Similarly for $e_k$ and $v_k$.
- There is $\epsilon > 0$ such that adding $\pm \epsilon (+1, -1, \ldots, ?1)$ to $x_P$ preserves feasibility.
- $x$ is the midpoint of $x + \epsilon (+1, -1, \ldots, ?1)_P$ and $x - \epsilon (+1, -1, \ldots, ?1)_P$, so $x$ is not a vertex.
The analogous statement holds for the perfect matching LP above, by an essentially identical proof.
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When a bipartite graph is complete and has the same number of nodes on either side, it can be equivalently phrased as a property of matrices.
Related Fact: Birkhoff Von-Neumann Theorem

\[ \sum_{e \in \delta(v)} x_e = 1, \quad \forall v \in V. \]
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- The analogous statement holds for the perfect matching LP above, by an essentially identical proof.
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Birkhoff Von-Neumann Theorem

The set of \( n \times n \) doubly stochastic matrices is the convex hull of \( n \times n \) permutation matrices.

\[
\begin{pmatrix}
0.5 & 0.5 \\
0.5 & 0.5
\end{pmatrix}
= 0.5 \begin{pmatrix}1 & 0 \\
0 & 1\end{pmatrix} + 0.5 \begin{pmatrix}0 & 1 \\
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By Caratheodory’s theorem, we can express every doubly stochastic matrix as a convex combination of $n^2 + 1$ permutation matrices.

We will see later: this decomposition can be computed efficiently!
We could have proved integrality of the bipartite matching LP using a more general tool.

**Definition**

A matrix $A$ is **Totally Unimodular** if every square submatrix has determinant 0, +1 or −1.

**Theorem**

If $A \in \mathbb{R}^{m \times n}$ is totally unimodular, and $b$ is an integer vector, then

$$\{x : Ax \leq b, x \geq 0\} \text{ has integer vertices.}$$
Total Unimodularity

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has integer vertices.

**Proof**

- Non-zero entries of vertex $x$ are solution of $A'x' = b'$ for some nonsingular square submatrix $A'$ and corresponding sub-vector $b'$
- Cramer's rule:

$$x'_i = \frac{\det(A'_i|b')}{\det A'}$$
Claim

The constraint matrix of the bipartite matching LP is totally unimodular.
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Proof

- $A_{ve} = 1$ if $e$ incident on $v$, and 0 otherwise.
- By induction on size of submatrix $A'$. Trivial for base case $k = 1$. 

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\sum_{e \in \delta(v)} x_e \leq 1, \quad \forall v \in V.
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- If $A'$ has all-zero column, then $\det A' = 0$
Total Unimodularity of Bipartite Matching

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- If \( A' \) has column with single 1, then holds by induction.
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- By induction on size of submatrix \( A' \). Trivial for base case \( k = 1 \).
- If \( A' \) has all-zero column, then \( \det A' = 0 \)
- If \( A' \) has column with single 1, then holds by induction.
- If all columns of \( A' \) have two 1’s,
  - Partition rows (vertices) into \( L \) and \( R \)
  - Sum of rows \( L \) is \((1, 1, \ldots, 1)\), similarly for \( R \)
  - \( A' \) is singular, so \( \det A' = 0 \).
Outline

1. Introduction
2. Shortest Path
4. Bipartite Matching
5. Total Unimodularity
6. Duality of Bipartite Matching and its Consequences
7. Spanning Trees
8. Flows
9. Max Cut
Primal and Dual LPs

**Primal LP**
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\begin{align*}
\text{max } & \sum_{e \in E} w_e x_e \\
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& x_e \geq 0, \quad \forall e \in E.
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**Dual LP**
\[
\begin{align*}
\text{min } & \sum_{v \in V} y_v \\
\text{s.t. } & y_u + y_v \geq w_e, \quad \forall e = (u, v) \in E. \\
& y_v \geq 0, \quad \forall v \in V.
\end{align*}
\]

- **Primal interpretation:** Player 1 looking to build a set of projects
  - Each edge \( e \) is a project generating “profit” \( w_e \)
  - Each project \( e = (u, v) \) needs two resources, \( u \) and \( v \)
  - Each resource can be used by at most one project at a time
  - Must choose a profit-maximizing set of projects

- **Dual interpretation:** Player 2 looking to buy resources
  - Offer a price \( y_v \) for each resource.
  - Prices should incentivize player 1 to sell resources
  - Want to pay as little as possible.
Primal and Dual LPs

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Vertex Cover Interpretation

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\begin{align*}
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When edge weights are 1, binary solutions to dual are vertex covers.

**Definition**

\(C \subseteq V\) is a **vertex cover** if every \(e \in E\) has at least one endpoint in \(C\).
## Vertex Cover Interpretation

**Primal LP**

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When edge weights are 1, binary solutions to dual are vertex covers.

**Definition**

\( C \subseteq V \) is a **vertex cover** if every \( e \in E \) has at least one endpoint in \( C \).

- Dual is a relaxation of the minimum vertex cover problem for bipartite graphs.
- By weak duality: \( \text{min-vertex-cover} \geq \text{max-cardinality-matching} \)
König’s Theorem

In a bipartite graph, the cardinality of the maximum matching is equal to the cardinality of the minimum vertex cover.

i.e. the dual LP has an optimal integral solution
Let $M(G)$ be a max cardinality of a matching in $G$

Let $C(G)$ be min cardinality of a vertex cover in $G$

We already proved that $M(G) \leq C(G)$

We will prove $C(G) \leq M(G)$ by induction on number of nodes in $G$. 

Note: Could have proved the same using total unimodularity
Let \( y \) be an optimal dual, and \( v \) a vertex with \( y_v > 0 \).
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- $M(G \setminus v) = M(G) - 1$
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Consequences of König’s Theorem

- Vertex covers can serve as a certificate of optimality for bipartite matchings, and vice versa.
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Like maximum cardinality matching, minimum vertex cover in bipartite graphs can be formulated as an LP, and solved in polynomial time.
Consequences of König’s Theorem

- Vertex covers can serve as a certificate of optimality for bipartite matchings, and vice versa.
- Like maximum cardinality matching, minimum vertex cover in bipartite graphs can be formulated as an LP, and solved in polynomial time.
- The same is true for the maximum independent set problem in bipartite graphs.
  - $C'$ is a vertex cover iff $V \setminus C'$ is an independent set.
The Minimum Cost Spanning Tree Problem

Given a connected undirected graph \( G = (V, E) \), and costs \( c_e \) on edges \( e \), find a minimum cost spanning tree of \( G \).

- **Spanning Tree**: an acyclic set of edges connecting every pair of nodes
- When graph is disconnected, can search for min-cost spanning forest instead
- We use \( n \) and \( m \) to denote \( |V| \) and \( |E| \), respectively.
Kruskal’s Algorithm

The minimum spanning tree problem can be solved efficiently by a simple greedy algorithm.

**Kruskal’s algorithm**

1. $T \leftarrow \emptyset$
2. Sort edges in increasing order of cost
3. For each edge $e$ in order
   - if $T \cup e$ is acyclic, add $e$ to $T$. 

Proof of correctness is via a simple exchange argument.

Generalizes to Matroids
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MST Linear Program

**MST LP**

minimize \( \sum_{e \in E} c_e x_e \)

subject to \( \sum_{e \in E} x_e = n - 1 \)

\( \sum_{e \subseteq X} x_e \leq |X| - 1, \text{ for } X \subseteq V. \)

\( x_e \geq 0, \text{ for } e \in E. \)

Theorem

The feasible region of the above LP is the convex hull of spanning trees.

Proof by finding a dual solution with cost matching the output of Kruskal's algorithm (on board)

Generalizes to Matroids

Note: this LP has an exponential (in \( n \)) number of constraints.
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- Note: this LP has an exponential (in \( n \)) number of constraints
A separation oracle for a linear program with feasible set $\mathcal{P} \subseteq \mathbb{R}^m$ is an algorithm which takes as input $x \in \mathbb{R}^m$, and either certifies that $x \in \mathcal{P}$ or identifies a violated constraint.
Solving the MST Linear Program

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A separation oracle for a linear program with feasible set $\mathcal{P} \subseteq \mathbb{R}^m$ is an algorithm which takes as input $x \in \mathbb{R}^m$, and either certifies that $x \in \mathcal{P}$ or identifies a violated constraint.

Theorem

A linear program with a polynomial number of variables is solvable in polynomial time if and only if it admits a polynomial time separation oracle (modulo some technicalities)

Follows from the ellipsoid method, which we will see next week.
Solving the MST Linear Program

Primal LP

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\begin{align*}
\text{minimize} & \quad \sum_{e \in E} c_e x_e \\
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- Given \( x \in \mathbb{R}^m \), separation oracle must find a violated constraint if one exists
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- Given \( x \in \mathbb{R}^m \), separation oracle must find a violated constraint if one exists
- Reduces to finding \( X \subset V \) with \( \sum_{e \subseteq X} x_e > |X| - 1 \), if one exists
  - Equivalently \( \frac{1 + \sum_{e \subseteq X} x_e}{|X|} > 1 \)
Solving the MST Linear Program

Primal LP

minimize \( \sum_{e \in E} c_e x_e \)
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Solving the MST Linear Program

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We will see how to do this efficiently later in the class, since \( \frac{1 + \sum_{e \subseteq X} x_e}{|X|} \) is a supermodular function of the set \( X \).
The LP formulation of spanning trees has many applications. We will look at one contrived yet simple application that shows the flexibility enabled by polyhedral formulation.

**Fault-Tolerant MST**

- Your tree is an overlay network on the internet used to transmit data.
- A hacker is looking to attack your tree, by knocking off one of the edges of the graph.
- You can foil the hacker by choosing a random tree.
- The hacker knows the algorithm you use, but not your random coins.
Fault-tolerant MST LP

\[
\begin{align*}
\text{minimize} & \quad \sum_{e \in E} c_e x_e \\
\text{subject to} & \quad \sum_{e \subseteq X} x_e \leq |X| - 1, \quad \text{for } X \subset V. \\
& \quad \sum_{e \in E} x_e = n - 1 \\
& \quad x_e \leq p, \quad \text{for } e \in E. \\
& \quad x_e \geq 0, \quad \text{for } e \in E.
\end{align*}
\]

- Above LP can be solved efficiently
- If feasible, can interpret resulting fractional spanning tree \( x \) as a recipe for a probability distribution over trees \( T \)
  - \( e \in T \) with probability \( x_e \)
  - Since \( x_e \leq p \), no edge is in the tree with probability more than \( p \).
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subject to \[ \sum_{e \subseteq X} x_e \leq |X| - 1, \quad \text{for } X \subset V. \]
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- Given feasible solution \( x \), such a probability distribution exists!
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- Given feasible solution \( x \), such a probability distribution exists!
- \( x \) is in the (original) MST polytope
- Caratheodory’s theorem: \( x \) is a convex combination of \( m + 1 \) vertices of MST polytope
- By integrality of MST polytope: \( x \) is the “expectation” of a probability distribution over spanning trees.
Given feasible solution $x$, such a probability distribution exists!

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- Caratheodory’s theorem: $x$ is a convex combination of $m + 1$ vertices of MST polytope
- By integrality of MST polytope: $x$ is the “expectation” of a probability distribution over spanning trees.

Consequence of Ellipsoid algorithm: can compute such a decomposition of $x$ efficiently!
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**The Maximum Flow Problem**

Given a directed graph $G = (V, E)$ with capacities $u_e$ on edges $e$, a source node $s$, and a sink node $t$, find a maximum flow from $s$ to $t$ respecting the capacities.

\[
\begin{align*}
\text{maximize} \quad & \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e \\
\text{subject to} \quad & \sum_{e \in \delta^-(v)} x_e = \sum_{e \in \delta^+(v)} x_e, \quad \text{for } v \in V \setminus \{s, t\}. \\
& x_e \leq u_e, \quad \text{for } e \in E. \\
& x_e \geq 0, \quad \text{for } e \in E.
\end{align*}
\]

Can be computed either by solving the LP, or by a combinatorial algorithm such as Ford Fulkerson.
Primal LP

\[
\begin{align*}
\text{max} & \quad \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e \\
\text{s.t.} & \quad \sum_{e \in \delta^-(v)} x_e = \sum_{e \in \delta^+(v)} x_e, \quad \forall v \in V \setminus \{s,t\} \\
& \quad x_e \leq u_e, \quad \forall e \in E. \\
& \quad x_e \geq 0, \quad \forall e \in E.
\end{align*}
\]

Dual LP (Simplified)

\[
\begin{align*}
\text{min} & \quad \sum_{e \in E} u_e z_e \\
\text{s.t.} & \quad \sum_{e \in P} y_u - \sum_{e \in P} y_v \leq z_e, \quad \forall e = (u,v) \in E. \\
& \quad y_s = 0 \\
& \quad y_t = 1 \\
& \quad z_e \geq 0, \quad \forall e \in E.
\end{align*}
\]

- Dual solution describes fraction $z_e$ of each edge to fractionally cut
Primal LP

\[
\begin{align*}
\max & \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e \\
\text{s.t.} & \sum_{e \in \delta^-(v)} x_e = \sum_{e \in \delta^+(v)} x_e, \quad \forall v \in V \setminus \{s, t\} \\
& x_e \leq u_e, \quad \forall e \in E. \\
& x_e \geq 0, \quad \forall e \in E.
\end{align*}
\]

Dual solution describes fraction \(z_e\) of each edge to fractionally cut

Dual constraints require that at least 1 edge is cut on every path from \(s\) to \(t\).

- \(\sum_{(u, v) \in P} z_{uv} \geq \sum_{(u, v) \in P} y_v - y_u = y_t - y_s = 1\)

Dual LP (Simplified)

\[
\begin{align*}
\min & \sum_{e \in E} u_e z_e \\
\text{s.t.} & y_v - y_u \leq z_e, \quad \forall e = (u, v) \in E. \\
& y_s = 0 \\
& y_t = 1 \\
& z_e \geq 0, \quad \forall e \in E.
\end{align*}
\]
Every integral $s - t$ cut is feasible.
Primal LP

\[
\max \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e \\
\text{s.t.} \sum_{e \in \delta^+(v)} x_e = \sum_{e \in \delta^-(v)} x_e, \quad \forall v \in V \setminus \{s, t\} \\
x_e \leq u_e, \quad \forall e \in E. \\
x_e \geq 0, \quad \forall e \in E.
\]

Dual LP (Simplified)

\[
\min \sum_{e \in E} u_e z_e \\
\text{s.t.} \sum_{e \in E} u_e z_e \leq z_e, \quad \forall e = (u, v) \in E. \\
y_s = 0 \\
y_t = 1 \\
z_e \geq 0, \quad \forall e \in E.
\]

- Every integral \(s - t\) cut is feasible.
- By weak duality: max flow \(\leq\) minimum cut
Every integral $s - t$ cut is feasible.

By weak duality: max flow $\leq$ minimum cut

Ford-Fulkerson shows that max flow $= \min$ cut
  i.e. dual has integer optimal
Primal LP

\[
\begin{align*}
\text{max} & \quad \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e \\
\text{s.t.} & \quad \sum_{e \in \delta^-(v)} x_e = \sum_{e \in \delta^+(v)} x_e, \quad \forall v \in V \setminus \{s,t\} \\
& \quad x_e \leq u_e, \quad \forall e \in E. \\
& \quad x_e \geq 0, \quad \forall e \in E.
\end{align*}
\]

Dual LP (Simplified)

\[
\begin{align*}
\text{min} & \quad \sum_{e \in E} u_e z_e \\
\text{s.t.} & \quad y_v - y_u \leq z_e, \quad \forall e = (u,v) \in E. \\
& \quad y_s = 0 \\
& \quad y_t = 1 \\
& \quad z_e \geq 0, \quad \forall e \in E.
\end{align*}
\]

- Every integral \(s - t\) cut is feasible.
- By weak duality: \(\text{max flow} \leq \text{minimum cut}\)
- Ford-Fulkerson shows that \(\text{max flow} = \text{min cut}\)
  - i.e. dual has integer optimal
- Ford-Fulkerson also shows that there is an integral optimal flow when capacities are integer.
Generalizations of Max Flow

\[
\begin{align*}
\text{max} & \quad \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e \\
\text{s.t.} & \quad \sum_{e \in \delta^-(v)} x_e = \sum_{e \in \delta^+(v)} x_e, \quad \forall v \in V \setminus \{s, t\}.
\end{align*}
\]

Where \( x_e \leq u_e \), \( \forall e \in E \),

\( x_e \geq 0 \), \( \forall e \in E \).

Writing as an LP shows that many generalizations are also tractable.
Generalizations of Max Flow

\[
\max \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e
\]

s.t.
\[
\sum_{e \in \delta^-(v)} x_e = \sum_{e \in \delta^+(v)} x_e, \quad \forall v \in V \setminus \{s, t\}.
\]
\[
x_e \leq u_e, \quad \forall e \in E.
\]
\[
x_e \geq 0, \quad \forall e \in E.
\]

Writing as an LP shows that many generalizations are also tractable

- Lower and upper bound constraints on flow: \( l_e \leq x_e \leq u_e \)
Generalizations of Max Flow

\[
\begin{align*}
\text{max} & \quad \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e \\
\text{s.t.} & \quad \sum_{e \in \delta^-(v)} x_e = \sum_{e \in \delta^+(v)} x_e, \quad \forall v \in V \setminus \{s, t\}.
\end{align*}
\]

\[
x_e \leq u_e, \quad \forall e \in E.
\]

\[
x_e \geq 0, \quad \forall e \in E.
\]

Writing as an LP shows that many generalizations are also tractable

- Lower and upper bound constraints on flow: \( \ell_e \leq x_e \leq u_e \)
- Minimum cost flow of a certain amount \( r \)
  - Objective \( \min \sum_e c_e x_e \)
  - Additional constraint: \( \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e = r \)
Generalizations of Max Flow

\[
\max \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e
\]

s.t.
\[
\sum_{e \in \delta^-(v)} x_e = \sum_{e \in \delta^+(v)} x_e, \quad \forall v \in V \setminus \{s, t\}.
\]
\[
x_e \leq u_e, \quad \forall e \in E.
\]
\[
x_e \geq 0, \quad \forall e \in E.
\]

Writing as an LP shows that many generalizations are also tractable

- Lower and upper bound constraints on flow: \( l_e \leq x_e \leq u_e \)
- Minimum cost flow of a certain amount \( r \)
  - Objective \( \min \sum e c_e x_e \)
  - Additional constraint: \( \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e = r \)
- Multiple commodities sharing the network
Generalizations of Max Flow

\[
\text{max } \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e \\
\text{s.t.} \\
\sum_{e \in \delta^-(v)} x_e = \sum_{e \in \delta^+(v)} x_e, \quad \forall v \in V \setminus \{s, t\}.
\]
\[
x_e \leq u_e, \quad \forall e \in E.
\]
\[
x_e \geq 0, \quad \forall e \in E.
\]

Writing as an LP shows that many generalizations are also tractable

- Lower and upper bound constraints on flow: \( l_e \leq x_e \leq u_e \)
- Minimum cost flow of a certain amount \( r \)
  - Objective \( \min \sum_e c_e x_e \)
  - Additional constraint: \( \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e = r \)
- Multiple commodities sharing the network
- ...
Minimum Congestion Flow

You are given a directed graph $G = (V, E)$ with congestion functions $c_e(\cdot)$ on edges $e$, a source node $s$, a sink node $t$, and a desired flow amount $r$. Find a minimum average congestion flow from $s$ to $t$.

\[
\begin{align*}
\text{minimize} & \quad \sum_e x_e c_e(x_e) \\
\text{subject to} & \quad \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e = r \\
& \quad \sum_{e \in \delta^-(v)} x_e = \sum_{e \in \delta^+(v)} x_e, \quad \text{for } v \in V \setminus \{s, t\}.
\end{align*}
\]

$x_e \geq 0$, for $e \in E$.

When $c_e(\cdot)$ are polynomials with nonnegative co-efficients, e.g. $c_e(x) = a_e x^2 + b_e x + c_e$ with $a_e, b_e, c_e \geq 0$, this is a (non-linear) convex program.
Outline

1. Introduction
2. Shortest Path
4. Bipartite Matching
5. Total Unimodularity
6. Duality of Bipartite Matching and its Consequences
7. Spanning Trees
8. Flows
9. Max Cut
The Max Cut Problem

Given an undirected graph $G = (V, E)$, find a partition of $V$ into $(S, V \setminus S)$ maximizing number of edges with exactly one end in $S$.

maximize $\sum_{(i,j) \in E} \frac{1-x_i x_j}{2}$

subject to $x_i \in \{-1, 1\}$, for $i \in V$. 
The Max Cut Problem

Given an undirected graph $G = (V, E)$, find a partition of $V$ into $(S, V \setminus S)$ maximizing number of edges with exactly one end in $S$.

\[
\begin{align*}
\text{maximize} & \quad \sum_{(i,j) \in E} \frac{1-x_i x_j}{2} \\
\text{subject to} & \quad x_i \in \{-1, 1\}, \quad \text{for } i \in V.
\end{align*}
\]

Instead of requiring $x_i$ to be on the 1 dimensional sphere, we relax and permit it to be in the $n$-dimensional sphere.

Vector Program relaxation

\[
\begin{align*}
\text{maximize} & \quad \sum_{(i,j) \in E} \frac{1-\vec{v}_i \cdot \vec{v}_j}{2} \\
\text{subject to} & \quad ||\vec{v}_i||_2 = 1, \quad \text{for } i \in V. \\
& \quad \vec{v}_i \in \mathbb{R}^n, \quad \text{for } i \in V.
\end{align*}
\]
Recall: A symmetric $n \times n$ matrix $Y$ is PSD iff $Y = V^T V$ for $n \times n$ matrix $V$
Equivalently: PSD matrices encode pairwise dot products of columns of $V$
When diagonal entries of $Y$ are 1, $V$ has unit length columns
Recall: $Y$ and $V$ can be recovered from each other efficiently
Recall: A symmetric $n \times n$ matrix $Y$ is PSD iff $Y = V^TV$ for $n \times n$ matrix $V$

Equivalently: PSD matrices encode pairwise dot products of columns of $V$

When diagonal entries of $Y$ are 1, $V$ has unit length columns

Recall: $Y$ and $V$ can be recovered from each other efficiently

**Vector Program relaxation**

$$
\begin{align*}
\text{maximize} & \quad \sum_{(i,j) \in E} \frac{1 - \vec{v}_i \cdot \vec{v}_j}{2} \\
\text{subject to} & \quad \|\vec{v}_i\|_2 = 1, \quad \text{for } i \in V. \\
& \quad \vec{v}_i \in \mathbb{R}^n, \quad \text{for } i \in V.
\end{align*}
$$

**SDP Relaxation**

$$
\begin{align*}
\text{maximize} & \quad \sum_{(i,j) \in E} \frac{1 - Y_{ij}}{2} \\
\text{subject to} & \quad Y_{ii} = 1, \quad \text{for } i \in V. \\
& \quad Y \in S^n_+
\end{align*}
$$
**SDP Relaxation**

maximize \[ \sum_{(i,j) \in E} \frac{1-Y_{ij}}{2} \]

subject to \[ Y_{ii} = 1, \quad \text{for } i \in V. \]
\[ Y \in S^n_+ \]

**Randomized Algorithm for Max Cut**

1. Solve the SDP to get \( Y \succeq 0 \)
2. Decompose \( Y \) to \( VV^T \)
3. Pick a random vector \( r \) on the unit sphere
4. Place all nodes \( i \) with \( v_i \cdot r \geq 0 \) on one side of the cut, and all others on the other side

Lemma
Consequently, by linearity of expectation, expected number of edges cut is at least \( \frac{1}{2} \cdot \text{OPT} \).
SDP Relaxation

Maximize
\[ \sum_{(i,j) \in E} \frac{1 - Y_{ij}}{2} \]
subject to
\[ Y_{ii} = 1, \quad \text{for } i \in V. \]
\[ Y \in S^n_+ \]

Randomized Algorithm for Max Cut

1. Solve the SDP to get \( Y \succeq 0 \)
2. Decompose \( Y \) to \( VV^T \)
3. Pick a random vector \( r \) on the unit sphere
4. Place all nodes \( i \) with \( v_i \cdot r \geq 0 \) on one side of the cut, and all others on the other side

Lemma

The SDP cuts each edge with probability at least \( 0.878 \frac{1 - Y_{ij}}{2} \).

Consequently, by linearity of expectation, expected number of edges cut is at least \( 0.878 \cdot OPT \).
Lemma

The SDP cuts each edge with probability at least $0.878 \frac{1-Y_{ij}}{2}$.

We use the following fact:

Fact

For all angles $\theta \in [0, \pi]$, $\frac{\theta}{\pi} \geq 0.878 \cdot \frac{1}{2} (1 - \cos(\theta))$

to prove the Lemma on the board.