Instructor: Shaddin Dughmi
1. Linear Programming Basics
2. Duality and Its Interpretations
3. Properties of Duals
4. Weak and Strong Duality
5. Formal Proof of Strong Duality of LP
6. Consequences of Duality
7. More Examples of Duality
Outline

1. Linear Programming Basics
2. Duality and Its Interpretations
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7. More Examples of Duality
The forefather of convex optimization problems, and the most ubiquitous.

Developed by Kantorovich during World War II (1939) for planning the Soviet army’s expenditures and returns. Kept secret.

Discovered a few years later by George Dantzig, who in 1947 developed the simplex method for solving linear programs.

John von Neumann developed LP duality in 1947, and applied it to game theory.

Polynomial-time algorithms: Ellipsoid method (Khachiyan 1979), interior point methods (Karmarkar 1984).
minimize (or maximize) \[ c^\top x \]
subject to
\[ a_i^\top x \leq b_i, \quad \text{for } i \in C^1. \]
\[ a_i^\top x \geq b_i, \quad \text{for } i \in C^2. \]
\[ a_i^\top x = b_i, \quad \text{for } i \in C^3. \]

- Decision variables: \( x \in \mathbb{R}^n \)
- Parameters:
  - \( c \in \mathbb{R}^n \) defines the linear objective function
  - \( a_i \in \mathbb{R}^n \) and \( b_i \in \mathbb{R} \) define the \( i \)’th constraint.
maximize \( c^T x \)
subject to \( a_i^T x \leq b_i \), for \( i = 1, \ldots, m \).
\( x_j \geq 0 \), for \( j = 1, \ldots, n \).

Every LP can be transformed to this form

- minimizing \( c^T x \) is equivalent to maximizing \(-c^T x\)
- \( \geq \) constraints can be flipped by multiplying by \(-1\)
- Each equality constraint can be replaced by two inequalities
- Uconstrained variable \( x_j \) can be replaced by \( x_j^+ - x_j^- \), where both \( x_j^+ \) and \( x_j^- \) are constrained to be nonnegative.
Geometric View
Geometric View
A 2-D example

maximize \[ x_1 + x_2 \]
subject to \[ x_1 + 2x_2 \leq 2 \]
\[ 2x_1 + x_2 \leq 2 \]
\[ x_1, x_2 \geq 0 \]
Application: Optimal Production

- $n$ products, $m$ raw materials
- Every unit of product $j$ uses $a_{ij}$ units of raw material $i$
- There are $b_i$ units of material $i$ available
- Product $j$ yields profit $c_j$ per unit
- Facility wants to maximize profit subject to available raw materials

maximize $c^T x$

subject to $a_i^T x \leq b_i$, for $i = 1, \ldots, m$.
$x_j \geq 0$, for $j = 1, \ldots, n$. 
Terminology

- **Hyperplane**: The region defined by a linear equality
- **Halfspace**: The region defined by a linear inequality $a_i^T x \leq b_i$.
- **Polyhedron**: The intersection of a set of linear inequalities
  - Feasible region of an LP is a polyhedron
- **Polytope**: Bounded polyhedron
  - Equivalently: convex hull of a finite set of points
- **Vertex**: A point $x$ is a vertex of polyhedron $P$ if $\exists y \neq 0$ with $x + y \in P$ and $x - y \in P$
- **Face**: The intersection with $P$ of a hyperplane $H$ disjoint from the interior of $P
Basic Facts about LPs and Polyhedrons

Fact

Feasible regions of LPs (i.e. polyhedrons) are convex
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Fact
Feasible regions of LPs (i.e. polyhedrons) are convex

Fact
Set of optimal solutions of an LP is convex
  - In fact, a face of the polyhedron
  - intersection of $P$ with hyperplane $c^T x = OPT$
Basic Facts about LPs and Polyhedrons

Fact
Feasible regions of LPs (i.e. polyhedrons) are convex

Fact
Set of optimal solutions of an LP is convex
- In fact, a face of the polyhedron
- intersection of $P$ with hyperplane $c^\top x = OPT$

Fact
At a vertex, $n$ linearly independent constraints are satisfied with equality (a.k.a. tight)
Fact
An LP either has an optimal solution, or is unbounded or infeasible
Fundamental Theorem of LP
If an LP in standard form has an optimal solution, then it has a vertex optimal solution.

Proof
Assume not, and take a non-vertex optimal solution $x$ with the maximum number of tight constraints. There is $y \neq 0$ s.t. $x \pm y$ are feasible. $y$ is perpendicular to the objective function and the tight constraints at $x$. i.e. $c^\top y = 0$, and $a_i^\top y = 0$ whenever the $i$'th constraint is tight for $x$.

Can choose $y$ s.t. $y_j < 0$ for some $j$. Let $\alpha$ be the largest constant such that $x + \alpha y$ is feasible. Such an $\alpha$ exists. An additional constraint becomes tight at $x + \alpha y$, a contradiction.
Fundamental Theorem of LP
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Linear Programming Basics 10/40
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  - i.e. $c^T y = 0$, and $a_i^T y = 0$ whenever the $i$’th constraint is tight for $x$.
- Can choose $y$ s.t. $y_j < 0$ for some $j$
- Let $\alpha$ be the largest constant such that $x + \alpha y$ is feasible
  - Such an $\alpha$ exists
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Assume not, and take a non-vertex optimal solution \( x \) with the maximum number of tight constraints.

There is \( y \neq 0 \) s.t. \( x \pm y \) are feasible.

\( y \) is perpendicular to the objective function and the tight constraints at \( x \).

i.e. \( c^T y = 0 \), and \( a_i^T y = 0 \) whenever the \( i \)'th constraint is tight for \( x \).

Can choose \( y \) s.t. \( y_j < 0 \) for some \( j \).

Let \( \alpha \) be the largest constant such that \( x + \alpha y \) is feasible.

Such an \( \alpha \) exists.

An additional constraint becomes tight at \( x + \alpha y \), a contradiction.
Corollary

If an LP in standard form has an optimal solution, then there is an optimal solution with at most \( m \) non-zero variables.

\[
\begin{align*}
\text{maximize} & \quad c^T x \\
\text{subject to} & \quad a_i^T x \leq b_i, \quad \text{for } i = 1, \ldots, m. \\
& \quad x_j \geq 0, \quad \text{for } j = 1, \ldots, n.
\end{align*}
\]

- e.g. for optimal production with \( n \) products and \( m \) raw materials, there is an optimal plan with at most \( m \) products.
Linear Programming Duality

**Primal LP**

maximize \( c^T x \)
subject to \( Ax \leq b \)

**Dual LP**

minimize \( b^T y \)
subject to \( A^T y = c \)
\( y \geq 0 \)

- \( A \in \mathbb{R}^{m \times n} \), \( c \in \mathbb{R}^n \), \( b \in \mathbb{R}^m \)
- \( y_i \) is the **dual variable** corresponding to primal constraint \( A_i x \leq b_i \)
- \( A^T_j y \geq c_j \) is the **dual constraint** corresponding to primal variable \( x_j \)
Linear Programming Duality: Standard Form, and Visualization

Primal LP

maximize $c^T x$
subject to $Ax \leq b$
$x \geq 0$

Dual LP

minimize $y^T b$
subject to $A^T y \geq c$
$y \geq 0$
Linear Programming Duality: Standard Form, and Visualization

### Primal LP

- **Objective**: Maximize $c^T x$
- **Constraints**:
  - $Ax \leq b$
  - $x \geq 0$

### Dual LP

- **Objective**: Minimize $y^T b$
- **Constraints**:
  - $A^T y \geq c$
  - $y \geq 0$

#### Table

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Linear Programming Duality: Standard Form, and Visualization

**Primal LP**

maximize \( c^T x \)

subject to

\( Ax \leq b \)

\( x \geq 0 \)

**Dual LP**

minimize \( y^T b \)

subject to

\( A^T y \geq c \)

\( y \geq 0 \)

- \( y_i \) is the **dual variable** corresponding to primal constraint \( A_i x \leq b_i \)
- \( A^T_j y \geq c_j \) is the **dual constraint** corresponding to primal variable \( x_j \)

\[
\begin{array}{cccc|c}
   & x_1 & x_2 & x_3 & x_4 \\
 y_1 & a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\
 y_2 & a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\
 y_3 & a_{31} & a_{32} & a_{33} & a_{34} & b_3 \\
   & c_1 & c_2 & c_3 & c_4 \\
\end{array}
\]
Interpretation 1: Economic Interpretation

Recall the Optimal Production problem from last lecture

- $n$ products, $m$ raw materials
- Every unit of product $j$ uses $a_{ij}$ units of raw material $i$
- There are $b_i$ units of material $i$ available
- Product $j$ yields profit $c_j$ per unit
- Facility wants to maximize profit subject to available raw materials
## Primal LP

\[
\begin{align*}
\text{max} \quad & \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \text{for } i \in [m]. \\
& x_j \geq 0, \quad \text{for } j \in [n].
\end{align*}
\]
## Interpretação 1: Eficácia Econômica

### Primal LP

\[
\begin{align*}
\text{max} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \text{for } i \in [m]. \\
& \quad x_j \geq 0, \quad \text{for } j \in [n].
\end{align*}
\]

### Dual LP

\[
\begin{align*}
\text{min} & \quad \sum_{i=1}^{m} b_i y_i \\
\text{s.t.} & \quad \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad \text{for } j \in [n]. \\
& \quad y_i \geq 0, \quad \text{for } i \in [m].
\end{align*}
\]

Dual variable \( y_i \) is a proposed price per unit of raw material \( i \).

Dual price vector is feasible if facility has incentive to sell materials.

Buyer wants to spend as little as possible to buy materials.
Interpretation 1: Economic Interpretation

**Primal LP**
\[
\begin{align*}
\text{max} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \text{for } i \in [m]. \\
& \quad x_j \geq 0, \quad \text{for } j \in [n].
\end{align*}
\]

**Dual LP**
\[
\begin{align*}
\text{min} & \quad \sum_{i=1}^{m} b_i y_i \\
\text{s.t.} & \quad \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad \text{for } j \in [n]. \\
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\end{align*}
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$Duality$ $and$ $Its$ $Interpretations
Interpretation 1: Economic Interpretation

**Primal LP**

\[
\begin{align*}
\text{max} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \text{for } i \in [m]. \\
& \quad x_j \geq 0, \quad \text{for } j \in [n].
\end{align*}
\]

**Dual LP**

\[
\begin{align*}
\text{min} & \quad \sum_{i=1}^{m} b_i y_i \\
\text{s.t.} & \quad \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad \text{for } j \in [n]. \\
& \quad y_i \geq 0, \quad \text{for } i \in [m].
\end{align*}
\]

- Dual variable \( y_i \) is a proposed price per unit of raw material \( i \)
- Dual price vector is feasible if facility has incentive to sell materials
- Buyer wants to spend as little as possible to buy materials
Consider the simple LP from last lecture

\[
\begin{align*}
\text{maximize} & \quad x_1 + x_2 \\
\text{subject to} & \quad x_1 + 2x_2 \leq 2 \\
& \quad 2x_1 + x_2 \leq 2 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

- We found that the optimal solution was at \((\frac{2}{3}, \frac{2}{3})\), with an optimal value of \(\frac{4}{3}\).
Interpretation 2: Finding the Best Upperbound

Consider the simple LP from last lecture

\[
\begin{align*}
\text{maximize} & \quad x_1 + x_2 \\
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& \quad x_1, x_2 \geq 0 
\end{align*}
\]

We found that the optimal solution was at \((\frac{2}{3}, \frac{2}{3})\), with an optimal value of \(\frac{4}{3}\).

What if, instead of finding the optimal solution, we sought to find an upperbound on its value by combining inequalities?

- Each inequality implies an upper bound of 2
- Multiplying each by \(\frac{1}{3}\) and summing gives \(x_1 + x_2 \leq \frac{4}{3}\).
Interpretation 2: Finding the Best Upperbound

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- Multiplying each row $i$ by $y_i$ and summing gives the inequality

$$y^T Ax \leq y^T b$$
## Interpretation 2: Finding the Best Upperbound

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Multiplying each row $i$ by $y_i$ and summing gives the inequality

$$y^T Ax \leq y^T b$$

When $y^T A \geq c^T$, the right hand side of the inequality is an upper bound on $c^T x$ for every feasible $x$.

$$c^T x \leq y^T Ax \leq y^T b$$
Interpretation 2: Finding the Best Upperbound

Multiplying each row \( i \) by \( y_i \) and summing gives the inequality

\[
y^T A x \leq y^T b
\]

When \( y^T A \geq c^T \), the right hand side of the inequality is an upper bound on \( c^T x \) for every feasible \( x \).

\[
c^T x \leq y^T A x \leq y^T b
\]

The dual LP can be thought of as trying to find the best upperbound on the primal that can be achieved this way.
Apply force field \( c \) to a ball inside bounded polytope \( Ax \leq b \).
Interpretation 3: Physical Forces

- Apply force field \( c \) to a ball inside bounded polytope \( Ax \leq b \).
- Eventually, ball will come to rest against the walls of the polytope.
Apply force field \( c \) to a ball inside bounded polytope \( Ax \leq b \).

Eventually, ball will come to rest against the walls of the polytope.

Wall \( a_i x \leq b_i \) applies some force \(-y_i a_i\) to the ball.
Interpretation 3: Physical Forces

- Apply force field $c$ to a ball inside bounded polytope $Ax \leq b$.
- Eventually, ball will come to rest against the walls of the polytope.
- Wall $a_i x \leq b_i$ applies some force $-y_i a_i$ to the ball.
- Since the ball is still, $c^T = \sum_i y_i a_i = y^T A$. 
Interpretation 3: Physical Forces

- Apply force field $c$ to a ball inside bounded polytope $Ax \leq b$.
- Eventually, ball will come to rest against the walls of the polytope.
- Wall $a_i x \leq b_i$ applies some force $-y_i a_i$ to the ball.
- Since the ball is still, $c^T = \sum_i y_i a_i = y^T A$.
- Dual can be thought of as trying to minimize “work” $\sum_i y_i b_i$ to bring ball back to origin by moving polytope.
- We will see that, at optimality, only the walls adjacent to the ball push (Complementary Slackness).
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Duality is an Inversion

Primal LP

maximize \( c^T x \)
subject to
\( Ax \leq b \)
\( x \geq 0 \)

Dual LP

minimize \( b^T y \)
subject to
\( A^T y \geq c \)
\( y \geq 0 \)

Duality is an Inversion

Given a primal LP in standard form, the dual of its dual is itself.
Correspondance Between Variables and Constraints

**Primal LP**

\[
\begin{align*}
\text{max} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \text{for } i \in [m]. \\
& \quad x_j \geq 0, \quad \text{for } j \in [n].
\end{align*}
\]

**Dual LP**

\[
\begin{align*}
\text{min} & \quad \sum_{i=1}^{m} b_i y_i \\
\text{s.t.} & \quad \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad \text{for } j \in [n]. \\
& \quad y_i \geq 0, \quad \text{for } i \in [m].
\end{align*}
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Correspondance Between Variables and Constraints

**Primal LP**

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\end{align*}
\]

**Dual LP**

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\text{min} & \quad \sum_{i=1}^{m} b_i y_i \\
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& \quad y_i \geq 0, \quad \text{for } i \in [m].
\end{align*}
\]

- The \(i\)’th primal constraint gives rise to the \(i\)’th dual variable \(y_i\).

Properties of Duals
Correspondance Between Variables and Constraints

**Primal LP**

\[
\begin{align*}
\text{max} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \text{for } i \in [m]. \\
& \quad x_j \geq 0, \quad \text{for } j \in [n].
\end{align*}
\]

**Dual LP**

\[
\begin{align*}
\text{min} & \quad \sum_{i=1}^{m} b_i y_i \\
\text{s.t.} & \quad \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad \text{for } j \in [n]. \\
& \quad y_i \geq 0, \quad \text{for } i \in [m].
\end{align*}
\]

- The \(i\)’th primal constraint gives rise to the \(i\)’th dual variable \(y_i\)
- The \(j\)’th primal variable \(x_j\) gives rise to the \(j\)’th dual constraint

---

Properties of Duals
Syntactic Rules

Primal LP

\[
\begin{align*}
\text{max} & \quad c^\top x \\
\text{s.t.} & \\
y_i : & \quad a_i x \leq b_i, \quad \text{for } i \in C_1. \\
y_i : & \quad a_i x = b_i, \quad \text{for } i \in C_2. \\
x_j & \geq 0, \quad \text{for } j \in D_1. \\
x_j & \in \mathbb{R}, \quad \text{for } j \in D_2.
\end{align*}
\]

Dual LP

\[
\begin{align*}
\text{min} & \quad b^\top y \\
\text{s.t.} & \\
x_j : & \quad \bar{a}_j^\top y \geq c_j, \quad \text{for } j \in D_1. \\
x_j : & \quad \bar{a}_j^\top y = c_j, \quad \text{for } j \in D_2. \\
y_i & \geq 0, \quad \text{for } i \in C_1. \\
y_i & \in \mathbb{R}, \quad \text{for } i \in C_2.
\end{align*}
\]

Rules of Thumb

- Loose constraint (i.e. inequality) \(\Rightarrow\) tight dual variable (i.e. nonnegative)
- Tight constraint (i.e. equality) \(\Rightarrow\) loose dual variable (i.e. unconstrained)
Outline

1. Linear Programming Basics
2. Duality and Its Interpretations
3. Properties of Duals
4. Weak and Strong Duality
5. Formal Proof of Strong Duality of LP
6. Consequences of Duality
7. More Examples of Duality
Weak Duality

**Primal LP**
- **Maximize**: \( c^T x \)
- **Subject to**: \( Ax \leq b \), \( x \geq 0 \)

**Dual LP**
- **Minimize**: \( b^T y \)
- **Subject to**: \( A^T y \geq c \), \( y \geq 0 \)

**Theorem (Weak Duality)**

*For every primal feasible \( x \) and dual feasible \( y \), we have \( c^T x \leq b^T y \).*

**Corollary**

- If primal and dual both feasible and bounded, \( \text{OPT}(\text{Primal}) \leq \text{OPT}(\text{Dual}) \)
- If primal is unbounded, dual is infeasible
- If dual is unbounded, primal is infeasible
Theorem (Weak Duality)

For every primal feasible $x$ and dual feasible $y$, we have $c^T x \leq b^T y$.

Corollary

If $x$ is primal feasible, and $y$ is dual feasible, and $c^T x = b^T y$, then both are optimal.
Economic Interpretation
If selling the raw materials is more profitable than making any individual product, then total money collected from sale of raw materials would exceed profit from production.
Interpretation of Weak Duality

**Economic Interpretation**

If selling the raw materials is more profitable than making any individual product, then total money collected from sale of raw materials would exceed profit from production.

**Upperbound Interpretation**

Self explanatory
## Interpretation of Weak Duality

### Economic Interpretation
If selling the raw materials is more profitable than making any individual product, then total money collected from sale of raw materials would exceed profit from production.

### Upperbound Interpretation
Self explanatory

### Physical Interpretation
Work required to bring ball back to origin by pulling polytope is at least potential energy difference between origin and primal optimum.
Proof of Weak Duality

Primal LP

maximize \[ c^\top x \]
subject to \[ Ax \leq b \]
\[ x \geq 0 \]

Dual LP

minimize \[ b^\top y \]
subject to \[ A^\top y \geq c \]
\[ y \geq 0 \]

\[ c^\top x \leq y^\top Ax \leq y^\top b \]
### Theorem (Strong Duality)

*If either the primal or dual is feasible and bounded, then so is the other and \( OPT(Primal) = OPT(Dual) \).*
Economic Interpretation

Buyer can offer prices for raw materials that would make facility indifferent between production and sale.
Economic Interpretation

Buyer can offer prices for raw materials that would make facility indifferent between production and sale.

Upperbound Interpretation

The method of scaling and summing inequalities yields a tight upperbound on the primal optimal value.
Interpretation of Strong Duality

Economic Interpretation
Buyer can offer prices for raw materials that would make facility indifferent between production and sale.

Upperbound Interpretation
The method of scaling and summing inequalities yields a tight upperbound on the primal optimal value.

Physical Interpretation
There is an assignment of forces to the walls of the polytope that brings ball back to the origin without wasting energy.
Informal Proof of Strong Duality

Recall the physical interpretation of duality

We found a primal and dual solution that are equal in value!
Recall the physical interpretation of duality:
When ball is stationary at $x$, we expect force $c$ to be neutralized only by constraints that are tight. i.e. force multipliers $y \geq 0$ s.t.

- $y^\top A = c$
- $y_i (b_i - a_i x) = 0$
Informal Proof of Strong Duality

Recall the physical interpretation of duality:
When ball is stationary at $x$, we expect force $c$ to be neutralized only by constraints that are tight. i.e. force multipliers $y \geq 0$ s.t.

- $y^\top A = c$
- $y_i(b_i - a_i x) = 0$

$$y^\top b - c^\top x = y^\top b - y^\top A x = \sum_i y_i(b_i - a_i x) = 0$$

We found a primal and dual solution that are equal in value!
Outline

1. Linear Programming Basics
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5. Formal Proof of Strong Duality of LP
6. Consequences of Duality
7. More Examples of Duality
Separating Hyperplane Theorem

If $A, B \subseteq \mathbb{R}^n$ are disjoint convex sets, then there is a hyperplane separating them. That is, there is $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $a^T x \leq b$ for every $x \in A$ and $a^T y \geq b$ for every $y \in B$. Moreover, if both $A$ and $B$ are closed and at least one of them is compact, then there is a hyperplane strictly separating them (i.e. $a^T x < b$ for $x \in A$ and $a^T y > b$ for $y \in B$).
Definition

A convex cone is a convex subset of $\mathbb{R}^n$ which is closed under nonnegative scaling and convex combinations.

Definition

The convex cone generated by vectors $u_1, \ldots, u_m \in \mathbb{R}^n$ is the set of all nonnegative-weighted sums of these vectors (also known as conic combinations).

$$Cone(u_1, \ldots, u_m) = \left\{ \sum_{i=1}^{m} \alpha_i u_i : \alpha_i \geq 0 \forall i \right\}$$
The following follows from the separating hyperplane Theorem (try to prove it).

### Farkas’ Lemma

Let $C$ be the **convex cone** generated by vectors $u_1, \ldots, u_m \in \mathbb{R}^n$, and let $w \in \mathbb{R}^n$. Exactly one of the following is true:

- $w \in C$
- There is $z \in \mathbb{R}^n$ such that $z \cdot u_i \leq 0$ for all $i$, and $z \cdot w > 0$. 

Formal Proof of Strong Duality of LP 28/40
Equivalently: Theorem of the Alternative

Exactly one of the following is true for \( U = [u_1, \ldots, u_m] \):

- The system \( Uy = w, \ y \geq 0 \) has a solution
- The system \( U^Tz \leq 0, \ z^T w > 0 \) has a solution.
Given $v \in \mathbb{R}$, by Farkas’ Lemma exactly one of the following is true

1. The system
   \[
   \begin{pmatrix}
   A^T & 0 \\
   b^T & 1
   \end{pmatrix} w = \begin{pmatrix} c \\ v \end{pmatrix}, \quad w \geq 0
   \]
   has a solution.
   - Let $y \in \mathbb{R}_+^m$ and $\delta \in \mathbb{R}_+$ be such that $w = \begin{pmatrix} y \\ \delta \end{pmatrix}$
   - Implies dual is feasible and $OPT\text{(dual)} \leq v$

2. The system
   \[
   \begin{pmatrix}
   A & b \\
   0 & 1
   \end{pmatrix} z \leq 0, \quad z^T \begin{pmatrix} c \\ v \end{pmatrix} > 0
   \]
   has a solution.
   - Let $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, where $z_1 \in \mathbb{R}^n$ and $z_2 \in \mathbb{R}$ with $z_2 \leq 0$
   - When $z_2 \neq 0$, $x = -z_1/z_2$ is feasible and $c^T x > v$
   - When $z_2 = 0$, primal is either infeasible or unbounded, and dual is infeasible (prove it)
1. Linear Programming Basics
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Complementary Slackness

Primal LP

maximize \( c^T x \)
subject to
\( Ax \leq b \)
\( x \geq 0 \)

Dual LP

minimize \( y^T b \)
subject to
\( A^T y \geq c \)
\( y \geq 0 \)
### Complementary Slackness

#### Primal LP

- maximize $c^T x$
- subject to $Ax \leq b$
- $x \geq 0$

#### Dual LP

- minimize $y^T b$
- subject to $A^T y \geq c$
- $y \geq 0$

- Let $s_i = (b - Ax)_i$ be the $i$'th primal slack variable.
- Let $t_j = (A^T y - c)_j$ be the $j$'th dual slack variable.
Complementary Slackness

**Primal LP**

maximize \( c^T x \)
subject to \( Ax \leq b \)
\( x \geq 0 \)

**Dual LP**

minimize \( y^T b \)
subject to \( A^T y \geq c \)
\( y \geq 0 \)

- Let \( s_i = (b - Ax)_i \) be the \( i \)'th primal slack variable
- Let \( t_j = (A^T y - c)_j \) be the \( j \)'th dual slack variable

**Complementary Slackness**

\( x \) and \( y \) are optimal if and only if
- \( x_j t_j = 0 \) for all \( j = 1, \ldots, n \)
- \( y_i s_i = 0 \) for all \( i = 1, \ldots, m \)
Economic Interpretation

Given an optimal primal production vector $x$ and optimal dual offer prices $y$,

- Facility produces only products for which it is indifferent between sale and production.
- Only raw materials that are binding constraints on production are priced greater than 0.
Physical Interpretation

Only walls adjacent to the balls equilibrium position push back on it.
Proof of Complementary Slackness

Primal LP

\[
\begin{align*}
\text{maximize} \quad & c^\top x \\
\text{subject to} \quad & Ax \leq b \\
& x \geq 0
\end{align*}
\]

Dual LP

\[
\begin{align*}
\text{minimize} \quad & y^\top b \\
\text{subject to} \quad & A^\top y \geq c \\
& y \geq 0
\end{align*}
\]

Can equivalently rewrite LP using slack variables:

\[
y^\top b - c^\top x = y^\top (Ax + s) - (y^\top A - t^\top) x = y^\top s + t^\top x
\]

Gap between primal and dual objectives is 0 if and only if complementary slackness holds.
Proof of Complementary Slackness

Primal LP

maximize \( c^\top x \)
subject to \( Ax + s = b \)
\( x \geq 0 \)
\( s \geq 0 \)

Dual LP

minimize \( y^\top b \)
subject to \( A^\top y - t = c \)
\( y \geq 0 \)
\( t \geq 0 \)

- Can equivalently rewrite LP using slack variables
Proof of Complementary Slackness

<table>
<thead>
<tr>
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<th>Dual LP</th>
</tr>
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<tbody>
<tr>
<td>maximize</td>
<td>minimize</td>
</tr>
<tr>
<td>$c^\top x$</td>
<td>$y^\top b$</td>
</tr>
<tr>
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</tr>
<tr>
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<td>$A^\top y - t = c$</td>
</tr>
<tr>
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- Can equivalently rewrite LP using slack variables

$$y^\top b - c^\top x = y^\top (Ax + s) - (y^\top A - t^\top)x = y^\top s + t^\top x$$
Proof of Complementary Slackness

Primal LP

maximize \( c^\top x \)
subject to \( Ax + s = b \)
\( x \geq 0 \)
\( s \geq 0 \)

Dual LP

minimize \( y^\top b \)
subject to \( A^\top y - t = c \)
\( y \geq 0 \)
\( t \geq 0 \)

Can equivalently rewrite LP using slack variables

\[
y^\top b - c^\top x = y^\top (Ax + s) - (y^\top A - t^\top)x = y^\top s + t^\top x
\]

Gap between primal and dual objectives is 0 if and only if complementary slackness holds.
Recovering Primal from Dual

- Will encounter LPs where the dual is easier to solve than primal
- Complementary slackness allows us to recover the primal optimal from the dual optimal, and vice versa.

Primal LP (n variables, m + n constraints)

\[
\begin{align*}
\text{maximize} & \quad c^\top x \\
\text{subject to} & \quad Ax \leq b \\
& \quad x \geq 0
\end{align*}
\]

Dual LP (m variables, m + n constraints)

\[
\begin{align*}
\text{minimize} & \quad y^\top b \\
\text{subject to} & \quad A^\top y \geq c \\
& \quad y \geq 0
\end{align*}
\]

Let \( y \) be dual optimal. By non-degeneracy:

- Exactly \( m \) of the \( m + n \) dual constraints are tight at \( y \)
- Exactly \( n \) dual constraints are loose

\( n \) loose dual constraints impose \( n \) tight primal constraints

Assuming non-degeneracy, solving the linear equation yields a unique primal optimum solution \( x \).
Recovering Primal from Dual

- Will encounter LPs where the dual is easier to solve than primal
- Complementary slackness allows us to recover the primal optimal from the dual optimal, and vice versa.
  - Assuming non-degeneracy: At every vertex of primal [dual] there are exactly $\n \times \m$ tight constraints which are linearly independent.

Primal LP ($n$ variables, $\m + n$ constraints)

$$\text{maximize } c \top x$$
$$\text{subject to } Ax \leq b$$
$$x \geq 0$$

Dual LP ($m$ variables, $\m + n$ constraints)

$$\text{minimize } y \top b$$
$$\text{subject to } A \top y \geq c$$
$$y \geq 0$$

Let $y$ be dual optimal. By non-degeneracy:

- Exactly $\m$ of the $\m + n$ dual constraints are tight at $y$
- Exactly $n$ dual constraints are loose

$n$ loose dual constraints impose $n$ tight primal constraints

Assuming non-degeneracy, solving the linear equation yields a unique primal optimum solution $x$. 
Recovering Primal from Dual

- Will encounter LPs where the dual is easier to solve than primal
- Complementary slackness allows us to recover the primal optimal from the dual optimal, and vice versa.
  - Assuming non-degeneracy: At every vertex of primal [dual] there are exactly $n$ [$m$] tight constraints which are linearly independent.

### Primal LP

**Primal LP**

(\(n\) variables, \(m + n\) constraints)

- maximize \(c^T x\)
- subject to \(Ax \leq b\)
- \(x \geq 0\)

### Dual LP

**Dual LP**

(\(m\) variables, \(m + n\) constraints)

- minimize \(y^T b\)
- subject to \(A^T y \geq c\)
- \(y \geq 0\)
Recovering Primal from Dual

- Will encounter LPs where the dual is easier to solve than primal
- Complementary slackness allows us to recover the primal optimal from the dual optimal, and vice versa.
  - Assuming non-degeneracy: At every vertex of primal [dual] there are exactly \( n \) [\( m \)] tight constraints which are linearly independent.

Primal LP
\( (n \) variables, \( m + n \) constraints)\

\[
\begin{align*}
\text{maximize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b \\
& \quad x \geq 0
\end{align*}
\]

Dual LP
\( (m \) variables, \( m + n \) constraints)\

\[
\begin{align*}
\text{minimize} & \quad y^T b \\
\text{subject to} & \quad A^T y \geq c \\
& \quad y \geq 0
\end{align*}
\]

- Let \( y \) be dual optimal. By non-degeneracy:
  - Exactly \( m \) of the \( m + n \) dual constraints are tight at \( y \)
  - Exactly \( n \) dual constraints are loose
## Recovering Primal from Dual

- Will encounter LPs where the dual is easier to solve than primal.
- Complementary slackness allows us to recover the primal optimal from the dual optimal, and vice versa.
  - Assuming non-degeneracy: At every vertex of primal [dual] there are exactly \( n \) \([m]\) tight constraints which are linearly independent.

### Primal LP

\((n \text{ variables, } m + n \text{ constraints})\)

- maximize \( c^T x \)
- subject to \( Ax \leq b \)
- \( x \geq 0 \)

### Dual LP

\((m \text{ variables, } m + n \text{ constraints})\)

- minimize \( y^T b \)
- subject to \( A^T y \geq c \)
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- Let \( y \) be dual optimal. By non-degeneracy:
  - Exactly \( m \) of the \( m + n \) dual constraints are tight at \( y \)
  - Exactly \( n \) dual constraints are loose
  - \( n \) loose dual constraints impose \( n \) tight primal constraints

---

Consequences of Duality
Recovering Primal from Dual

- Will encounter LPs where the dual is easier to solve than primal
- Complementary slackness allows us to recover the primal optimal from the dual optimal, and vice versa.
  - Assuming non-degeneracy: At every vertex of primal [dual] there are exactly $n$ [m] tight constraints which are linearly independent.

**Primal LP**

$(n \text{ variables, } m + n \text{ constraints})$

- maximize $c^T x$
- subject to $Ax \leq b$
- $x \geq 0$

**Dual LP**

$(m \text{ variables, } m + n \text{ constraints})$

- minimize $y^T b$
- subject to $A^T y \geq c$
- $y \geq 0$

Let $y$ be dual optimal. By non-degeneracy:
- Exactly $m$ of the $m + n$ dual constraints are tight at $y$
- Exactly $n$ dual constraints are loose
- $n$ loose dual constraints impose $n$ tight primal constraints
  - Assuming non-degeneracy, solving the linear equation yields a unique primal optimum solution $x$. 
Sometimes, we want to examine how the optimal value of our LP changes with its parameters $c$ and $b$
Sensitivity Analysis

**Primal LP**

maximize \( c^T x \)
subject to \( Ax \leq b \)
\( x \geq 0 \)

**Dual LP**

minimize \( y^T b \)
subject to \( A^T y \geq c \)
\( y \geq 0 \)

Sometimes, we want to examine how the optimal value of our LP changes with its parameters \( c \) and \( b \)

**Sensitivity Analysis**

Let \( OPT = OPT(A, c, b) \) be the optimal value of the above LP. Let \( x \) and \( y \) be the primal and dual optima.

- \( \frac{\partial OPT}{\partial c_j} = x_j \) when \( x \) is the unique primal optimum.
- \( \frac{\partial OPT}{\partial b_i} = y_i \) when \( y \) is the unique dual optimum.
Sensitivity Analysis

Primal LP

maximize \( c^T x \)
subject to \( Ax \leq b \)
\( x \geq 0 \)

Dual LP

minimize \( y^T b \)
subject to \( A^T y \geq c \)
\( y \geq 0 \)

Sometimes, we want to examine how the optimal value of our LP changes with its parameters \( c \) and \( b \)

Economic Interpretation of Sensitivity Analysis

- A small increase \( \delta \) in \( c_j \) increases profit by \( \delta \cdot x_j \)
- A small increase \( \delta \) in \( b_i \) increases profit by \( \delta \cdot y_i \)
  
  \( y_i \) measures the “marginal value” of resource \( i \) for production
Given a directed network $G = (V, E)$ where edge $e$ has length $\ell_e \in \mathbb{R}_+$, find the minimum cost path from $s$ to $t$. 

Interpretation of Dual

Stretch $s$ and $t$ as far apart as possible, subject to edge lengths.

More Examples of Duality
Primal LP

\[
\begin{align*}
& \text{min} & & \sum_{e \in E} \ell_e x_e \\
& \text{s.t.} & & \sum_{e \to v} x_e - \sum_{v \to e} x_e = \delta_v, \quad \forall v \in V. \\
& & & x_e \geq 0, \quad \forall e \in E.
\end{align*}
\]

Where \( \delta_v = -1 \) if \( v = s \), \( 1 \) if \( v = t \), and \( 0 \) otherwise.

Dual LP

\[
\begin{align*}
& \text{max} & & y_t - y_s \\
& \text{s.t.} & & y_v - y_u \leq \ell_e, \quad \forall (u, v) \in E.
\end{align*}
\]
Shortest Path

**Primal LP**

\[
\begin{align*}
\min & \quad \sum_{e \in E} \ell_e x_e \\
\text{s.t.} & \quad \sum_{e \to v} x_e - \sum_{v \to e} x_e = \delta_v, \quad \forall v \in V. \\
& \quad x_e \geq 0, \quad \forall e \in E.
\end{align*}
\]

Where \( \delta_v = -1 \) if \( v = s \), \( 1 \) if \( v = t \), and \( 0 \) otherwise.

**Dual LP**

\[
\begin{align*}
\max & \quad y_t - y_s \\
\text{s.t.} & \quad y_v - y_u \leq \ell_e, \quad \forall (u, v) \in E.
\end{align*}
\]

**Interpretation of Dual**

Stretch \( s \) and \( t \) as far apart as possible, subject to edge lengths.
Maximum Weighted Bipartite Matching

Set $B$ of buyers, and set $G$ of goods. Buyer $i$ has value $w_{ij}$ for good $j$, and interested in at most one good. Find maximum value assignment of goods to buyers.
# Maximum Weighted Bipartite Matching

## Primal LP

\[
\begin{align*}
\text{max} & \quad \sum_{i,j} w_{ij} x_{ij} \\
\text{s.t.} & \quad \sum_{j \in G} x_{ij} \leq 1, \quad \forall i \in B. \\
& \quad \sum_{i \in B} x_{ij} \leq 1, \quad \forall j \in G. \\
& \quad x_{ij} \geq 0, \quad \forall i \in B, j \in G.
\end{align*}
\]

## Dual LP

\[
\begin{align*}
\text{min} & \quad \sum_{i \in B} u_i + \sum_{j \in G} p_j \\
\text{s.t.} & \quad u_i + p_j \geq w_{ij}, \quad \forall i \in B, j \in G. \\
& \quad u_i \geq 0, \quad \forall i \in B. \\
& \quad p_j \geq 0, \quad \forall j \in G.
\end{align*}
\]

### Interpretation of Dual

- \(p_j\) is price of good \(j\)
- \(u_i\) is utility of buyer \(i\)

### Complementary Slackness:

- A buyer \(i\) only grabs goods \(j\) maximizing \(w_{ij} - p_j\)
- Only fully assigned goods have non-zero price
- A buyer with nonzero utility must receive an item
### Maximum Weighted Bipartite Matching

#### Primal LP

\[
\text{max} \quad \sum_{i,j} w_{ij} x_{ij}
\]

s.t.

\[
\sum_{j \in G} x_{ij} \leq 1, \quad \forall i \in B.
\]

\[
\sum_{i \in B} x_{ij} \leq 1, \quad \forall j \in G.
\]

\[
x_{ij} \geq 0, \quad \forall i \in B, j \in G.
\]

#### Dual LP

\[
\text{min} \quad \sum_{i \in B} u_i + \sum_{j \in G} p_j
\]

s.t.

\[
u_i + p_j \geq w_{ij}, \quad \forall i \in B, j \in G.
\]

\[
u_i \geq 0, \quad \forall i \in B.
\]

\[
p_j \geq 0, \quad \forall j \in G.
\]

### Interpretation of Dual

- \( p_j \) is price of good \( j \)
- \( u_i \) is utility of buyer \( i \)

Complementary Slackness:

- A buyer \( i \) only grabs goods \( j \) maximizing \( w_{ij} - p_j \)
- Only fully assigned goods have non-zero price
- A buyer with nonzero utility must receive an item
Two players, row and column

Game described by matrix $A$

When row player plays pure strategy $i$ and column player plays pure strategy $j$, row player pays column player $A_{ij}$

### Rock-Paper-Scissors

<table>
<thead>
<tr>
<th></th>
<th>$R$</th>
<th>$P$</th>
<th>$S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td>0</td>
<td>1</td>
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2-Player Zero-Sum Games

Rock-Paper-Scissors

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- Two players, row and column
- Game described by matrix $A$
- When row player plays pure strategy $i$ and column player plays pure strategy $j$, row player pays column player $A_{ij}$
- **Mixed Strategy**: distribution over pure strategies
2-Player Zero-Sum Games

Rock-Paper-Scissors

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- Assume players know each other’s mixed strategies but not coin flips
2-Player Zero-Sum Games

- Assume row player moves first with distribution $y \in \Delta_m$
- Loss as a function of column’s strategy given by $y^T A$
- A best response by column is pure strategy $j$ maximizing $(y^T A)_j$

<table>
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<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
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<td>$y_3$</td>
<td>$a_{31}$</td>
<td>$a_{32}$</td>
<td>$a_{33}$</td>
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**Row Moves First**

\[
\begin{align*}
\min & \quad \max_j (y^\top A)_j \\
\text{s.t.} & \quad \sum_{i=1}^m y_i = 1 \\
& \quad y \geq 0
\end{align*}
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Row Moves First

\[
\begin{align*}
\text{min} & \quad u \\
\text{s.t.} & \quad u\mathbf{1} - y^T A \geq \mathbf{0} \\
& \quad \sum_{i=1}^{m} y_i = 1 \\
& \quad y \geq \mathbf{0}
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- Similarly when column moves first

\[
\begin{align*}
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& & \sum_{i=1}^{m} y_i & = 1 \\
& & y & \geq \vec{0}
\end{align*}
\]

\[
\begin{align*}
\text{Column Moves First} & : & \max \quad & v \\
\text{s.t.} & : & v\vec{1} - A x & \leq \vec{0} \\
& & \sum_{j=1}^{n} x_j & = 1 \\
& & x & \geq \vec{0}
\end{align*}
\]

These two optimization problems are LP Duals!
2-Player Zero-Sum Games

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Row Moves First

$$\begin{align*}
\text{min} & \quad u \\
\text{s.t.} & \quad u\mathbf{1} - y^T A \geq \mathbf{0} \\
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& \quad y \geq \mathbf{0}
\end{align*}$$

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$$\begin{align*}
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These two optimization problems are LP Duals!
Weak Duality

- \( u^* \geq v^* \)
- Zero sum games have a second mover advantage (weakly)
Duality and Zero Sum Games

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Strong Duality (Minimax Theorem)
- \( u^* = v^* \)
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**Complementary Slackness**
- \( x^* \) randomizes over pure best responses to \( y^* \), and vice versa.
Saddle Point Interpretation

Consider the matching pennies game

<table>
<thead>
<tr>
<th></th>
<th>$H$</th>
<th>$T$</th>
</tr>
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<tbody>
<tr>
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- Unique equilibrium: each player randomizes uniformly
- If row player deviates, he pays out more
- If column player deviates, he gets paid less
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