General Instructions  The following assignment is meant to be challenging. Feel free to discuss with fellow students, though please write up your solutions independently and acknowledge everyone you discussed the homework with on your writeup. I also expect that you will not attempt to consult outside sources, on the Internet or otherwise, for solutions to any of these homework problems.

Several of these problems are drawn from Boyd and Vendenberghe. I have divided the problems into three sets: easy, medium, and difficult. Finally, whenever a question asks you to “show” or “prove” a claim, please provide a formal mathematical proof.

1  Easy Problems

Problem 1. (4 points)
B&V Exercise 5.1.

Problem 2. (3 points)
B&V Exercise 5.3.

Problem 3. (3 points)
B&V Exercise 5.22.

Problem 4. (3 points)

Problem 5. (4 points)
B&V Exercise 5.27.

2  Medium Problems

Problem 6. (4 points)
B&V Exercise 5.13.
Problem 7. (4 points)
B&V Exercise 5.18.

Problem 8. (4 points)
B&V Exercise 5.42.

Problem 9. (4 points)
One of your fellow students asked a poignant question in class: What happens if we augment the shortest path LP with the constraints $x_e \leq 1$ for each edge $e$ in the graph? One might hope that doing so would prevent cycles in the path, and lead to an LP which computes the shortest simple path from $s$ to $t$. Show that this does not work — intuitively, the ability to select negative cycles “gets in the way” of finding the shortest path.
(Hint: exhibit a weighted graph $G$ where the optimal solution of this augmented LP is integral, and consists of a suboptimal path $P$ and a negative cycle $C$)

Problem 10. (4 points)
Let $G = (L \cup R, E)$ be a bipartite graph with left hand side nodes $L$, right hand side nodes $R$, and $|L| = |R| = n$. As shorthand, for $S \subseteq L$ we use $\mathcal{N}(S) = \{v \in R : \exists u \in S \text{ s.t. } (u, v) \in E\}$ to denote the neighbors of $S$. Use König’s theorem from class to prove Hall’s theorem: $G$ has a perfect matching (i.e. a matching with $n$ edges) if and only if $|\Gamma(S)| \geq |S|$ for all $S \subseteq L$.

3 Difficult Problems

Problem 11. (5 points)
B&V Exercise 5.37

Problem 12. (6 points)
B&V Exercise 5.39

Problem 13. (16 points)
In this problem we will establish some basic properties of polyhedra and linear programs. Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a nonempty polyhedron, let $V \subseteq \mathcal{P}$ be the vertices of $\mathcal{P}$, let $OPT(c) = \max_{x \in \mathcal{P}} c^T x$ for each $c \in \mathbb{R}^n$, and let $OPTSOL(c) = \text{argmax}_{x \in \mathcal{P}} c^T x$ for each $c \in \mathbb{R}^n$.

a (2 points). Fix $x \in V$, and let $C_x = \{c \in \mathbb{R}^n : x \in OPTSOL(c)\}$. Note that $C_x$ is the family of objectives for which $x$ is an optimal solution. Show that $C_x$ is a convex cone.

b (4 points). Fix $x \in V$. Show that there is a linear objective $c \in \mathbb{R}^n$ such that $OPTSOL(c) = \{x\}$.

c (4 points). Let $C_\infty := \{c \in \mathbb{R}^n : OPT(c) = \infty\}$. Note that $C_\infty$ is the family of objectives for which the LP is unbounded. Is $C_\infty$ a cone? Is it a convex cone? Prove or disprove.
\textbf{Problem 14. (6 points)}

Recall the single-source shortest path problem presented in class, where $m$ denotes the number of edges of a graph and $n$ denotes the number of vertices. Recall that the Bellman-Ford algorithm runs in time $O(mn)$, and computes the shortest path from the given source $s$ to every other vertex in the graph. Moreover, the classical Dijkstra’s algorithm solves the same problem on graphs with nonnegative-weighted edges in time $O(m + n \log n)$. Show to combine both algorithms to solve the \textit{all-pairs shortest path} problem, in graphs with arbitrary edge weights but no negative cycles, in time $O(mn + n^2 \log n)$ time. In this problem, you must compute a shortest path between every pair of vertices in the graph.