Convex Sets

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1 Convex sets, Affine sets, and Cones

2 Examples of Convex Sets

3 Convexity-Preserving Operations

4 Separation Theorems
Convex Sets

A set $S \subseteq \mathbb{R}^n$ is convex if the line segment between any two points in $S$ lies in $S$. i.e. if $x, y \in S$ and $\theta \in [0, 1]$, then $\theta x + (1 - \theta) y \in S$. 

![Convex Sets Diagram](image-url)
**Convex Sets**

A set $S \subseteq \mathbb{R}^n$ is **convex** if the line segment between any two points in $S$ lies in $S$. i.e. if $x, y \in S$ and $\theta \in [0, 1]$, then $\theta x + (1 - \theta) y \in S$.

**Equivalent Definition**

$S$ is convex if every convex combination of points in $S$ lies in $S$.

**Convex Combination**

- **Finite**: $y$ is a convex combination of $x_1, \ldots, x_k$ if $y = \theta_1 x_1 + \ldots + \theta_k x_k$, where $\theta_i \geq 0$ and $\sum_i \theta_i = 1$.
- **General**: expectation of probability measure on $S$. 

Convex sets, Affine sets, and Cones
Convex Sets

Convex Hull

The convex hull of $S \subseteq \mathbb{R}^n$ is the smallest convex set containing $S$.

- Intersection of all convex sets containing $S$
- The set of all convex combinations of points in $S$
Convex Sets

Convex Hull

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- Intersection of all convex sets containing \( S \)
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A set \( S \) is convex if and only if \( \text{convexhull}(S) = S \).
A set $S \subseteq \mathbb{R}^n$ is **affine** if the line passing through any two points in $S$ lies in $S$. i.e. if $x, y \in S$ and $\theta \in \mathbb{R}$, then $\theta x + (1 - \theta)y \in S$.

Obviously, affine sets are convex.
Affine Set

A set $S \subseteq \mathbb{R}^n$ is affine if the line passing through any two points in $S$ lies in $S$. i.e. if $x, y \in S$ and $\theta \in \mathbb{R}$, then $\theta x + (1 - \theta)y \in S$.

Obviously, affine sets are convex.

Equivalent Definition

$S$ is affine if every affine combination of points in $S$ lies in $S$.

Affine Combination

$y$ is an affine combination of $x_1, \ldots, x_k$ if $y = \theta_1 x_1 + \ldots + \theta_k x_k$, and $\sum_i \theta_i = 1$.

Generalizes convex combinations
Equivalent Definition II

$S$ is affine if and only if it is a shifted subspace

- i.e. $S = x_0 + V$, where $V$ is a linear subspace of $\mathbb{R}^n$.

- Any $x_0 \in S$ will do, and yields the same $V$.
- The dimension of $S$ is the dimension of subspace $V$. 

Convex sets, Affine sets, and Cones
Equivalent Definition II

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Equivalent Definition III

$S$ is affine if and only if it is the solution of a set of linear equations (i.e. the intersection of hyperplanes).

- i.e. $S = \{ x : Ax = b \}$ for some matrix $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. 
Affine Sets

Affine Hull

The affine hull of \( S \subseteq \mathbb{R}^n \) is the smallest affine set containing \( S \).

- Intersection of all affine sets containing \( S \)
- The set of all affine combinations of points in \( S \)
Affine Sets

Affine Hull

The affine hull of $S \subseteq \mathbb{R}^n$ is the smallest affine set containing $S$.

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A set $S$ is affine if and only if affinehull$(S) = S$. 
Affine Sets

Affine Hull
The affine hull of $S \subseteq \mathbb{R}^n$ is the smallest affine set containing $S$.
- Intersection of all affine sets containing $S$
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A set $S$ is affine if and only if $\text{affinehull}(S) = S$.

Affine Dimension
The affine dimension of a set is the dimension of its affine hull.
A set $K \subseteq \mathbb{R}^n$ is a cone if the ray from the origin through every point in $K$ is in $K$ i.e. if $x \in K$ and $\theta \geq 0$, then $\theta x \in K$.

Note: every cone contains 0.
A set $K \subseteq \mathbb{R}^n$ is a **cone** if the ray from the origin through every point in $K$ is in $K$ i.e. if $x \in K$ and $\theta \geq 0$, then $\theta x \in K$.

Note: every cone contains $0$.

**Special Cones**

- A **convex cone** is a cone that is convex
- A cone is **pointed** if whenever $x \in K$ and $x \neq 0$, then $-x \notin K$.
- We will mostly mention **proper** cones: convex, pointed, closed, and of full affine dimension.
Equivalent Definition

$K$ is a convex cone if every conic combination of points in $K$ lies in $K$.

Conic Combination

$y$ is a conic combination of $x_1, \ldots, x_k$ if $y = \theta_1 x_1 + \ldots + \theta_k x_k$, where $\theta_i \geq 0$. 
The conic hull of $K \subseteq \mathbb{R}^n$ is the smallest convex cone containing $K$.

- Intersection of all convex cones containing $K$
- The set of all conic combinations of points in $K$
Cones

Conic Hull

The conic hull of $K \subseteq \mathbb{R}^n$ is the smallest convex cone containing $K$

- Intersection of all convex cones containing $K$
- The set of all conic combinations of points in $K$

A set $K$ is a convex cone if and only if $\text{conichull}(K) = K$. 
A cone is polyhedral if it is the set of solutions to a finite set of homogeneous linear inequalities $Ax \leq 0$. 

**Polyhedral Cone**
Outline

1. Convex sets, Affine sets, and Cones
2. Examples of Convex Sets
3. Convexity-Preserving Operations
4. Separation Theorems
- Linear Subspace: Affine, Cone
- Hyperplane: Affine, cone if includes 0
- Halfspace: Cone if origin on boundary
- Line: Affine, cone if includes 0
- Ray: Cone if endpoint at 0
- Line segment
- Polyhedron: finite intersection of halfspaces

- Polytope: Bounded polyhedron
Nonnegative Orthant $\mathbb{R}^n_+$: Polyhedral cone

Simplex: convex hull of affinely independent points
  - Unit simplex: $x \succeq 0, \sum_i x_i \leq 1$
  - Probability simplex: $x \succeq 0, \sum_i x_i = 1$. 

Examples of Convex Sets 11/20
- Euclidean ball: \( \{ x : \| x - x_c \|_2 \leq r \} \) for center \( x_c \) and radius \( r \)
- Ellipsoid: \( \{ x : (x - x_c)^T P^{-1} (x - x_c) \leq 1 \} \) for symmetric \( P \succeq 0 \)
  - Equivalently: \( \{ x_c + Au : \| u \|_2 \leq 1 \} \) for some linear map \( A \)
Norm ball: \( \{ x : \| x - c \| \leq r \} \) for any norm \( \| \cdot \| \). 

The unit sphere for different metrics: \( \| x \|_p = 1 \) in \( \mathbb{R}^2 \).
Norm ball: \( \{ x : \| x - c \| \leq r \} \) for any norm \( \| . \| \)

The unit sphere for different metrics: \( \| x \|_{l_p} = 1 \) in \( \mathbb{R}^2 \).

Norm cone: \( \{ (x, r) : \| x \| \leq r \} \)

Cone of symmetric positive semi-definite matrices \( M \)
- Symmetric matrix \( A \succeq 0 \) iff \( x^T A x \geq 0 \) for all \( x \)
Outline

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The intersection of two convex sets is convex. This holds for the intersection of an infinite number of sets.

Examples

- Polyhedron: intersection of halfspaces
- PSD cone: intersection of linear inequalities $z^T A z \geq 0$, for all $z \in \mathbb{R}^n$. 
The intersection of two convex sets is convex. This holds for the intersection of an infinite number of sets.

Examples

- Polyhedron: intersection of halfspaces
- PSD cone: intersection of linear inequalities $z^T A z \geq 0$, for all $z \in \mathbb{R}^n$.

In fact, we will see that every closed convex set is the intersection of a (possibly infinite) set of halfspaces.
Affine Maps

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is an affine function (i.e. $f(x) = Ax + b$), then

- $f(S)$ is convex whenever $S \subseteq \mathbb{R}^n$ is convex
- $f^{-1}(T)$ is convex whenever $T \subseteq \mathbb{R}^m$ is convex

$$f(\theta x + (1 - \theta)y) = A(\theta x + (1 - \theta)y) + b$$

$$= \theta(Ax + b) + (1 - \theta)(Ay + b))$$

$$= \theta f(x) + (1 - \theta)f(y)$$
Examples

- An ellipsoid is image of a unit ball after an affine map
- A polyhedron $Ax \preceq b$ is inverse image of nonnegative orthant under $f(x) = b - Ax$
Perspective Function

Let $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be $P(x, t) = x/t$.

- $P(S)$ is convex whenever $S \subseteq \mathbb{R}^{n+1}$ is convex
- $P^{-1}(T)$ is convex whenever $T \subseteq \mathbb{R}^n$ is convex
Perspective Function

Let $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be $P(x, t) = x/t$.

- $P(S)$ is convex whenever $S \subseteq \mathbb{R}^{n+1}$ is convex
- $P^{-1}(T)$ is convex whenever $T \subseteq \mathbb{R}^n$ is convex

Generalizes to linear fractional functions $f(x) = \frac{Ax + b}{c^Tx + d}$

- Composition of perspective with affine.
Outline

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Separating Hyperplane Theorem

If $A, B \subseteq \mathbb{R}^n$ are disjoint convex sets, then there is a hyperplane weakly separating them. That is, there is $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $a^T x \leq b$ for every $x \in A$ and $a^T y \geq b$ for every $y \in B$. 
Separating Hyperplane Theorem (Strict Version)

If $A, B \subseteq \mathbb{R}^n$ are disjoint closed convex sets, and at least one of them is compact, then there is a hyperplane strictly separating them. That is, there is $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $a^\top x < b$ for every $x \in A$ and $a^\top y > b$ for every $y \in B$. 

Separating line

Separating axis
Farkas’ Lemma

Let $K$ be a closed convex cone and let $w \notin K$. There is $z \in \mathbb{R}^n$ such that $z^T x \geq 0$ for all $x \in K$, and $z^T w < 0$. 
Supporting Hyperplane

Supporting Hyperplane Theorem.

If $S \subseteq \mathbb{R}^n$ is a closed convex set and $y$ is on the boundary of $S$, then there is a hyperplane supporting $S$ at $y$. That is, there is $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $a^\top x \leq b$ for every $x \in S$ and $a^\top y = b$. 