CS675: Convex and Combinatorial Optimization
Fall 2014
Convex Functions

Instructor: Shaddin Dughmi
Outline

1. Convex Functions

2. Examples of Convex and Concave Functions

3. Convexity-Preserving Operations
A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if the line segment between any points on the graph of $f$ lies above $f$. i.e. if $x, y \in \mathbb{R}^n$ and $\theta \in [0, 1]$, then

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$
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- Inequality called **Jensen’s inequality** (basic form)
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- $f$ is convex iff its restriction to any line $\{x + tv : t \in \mathbb{R}\}$ is convex
- $f$ is strictly convex if inequality strict when $x \neq y$.
- Analogous definition when the domain of $f$ is a convex subset $D$ of $\mathbb{R}^n$
A function is $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is concave if $-f$ is convex. Equivalently:

- Line segment between any points on the graph of $f$ lies below $f$.
- If $x, y \in \mathbb{R}^n$ and $\theta \in [0, 1]$, then
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$f : \mathbb{R}^n \to \mathbb{R}$ is affine if it is both concave and convex. Equivalently:

- Line segment between any points on the graph of $f$ lies on the graph of $f$.
- $f(x) = a^T x + b$ for some $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. 
We will now look at some equivalent definitions of convex functions

**First Order Definition**

A differentiable $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex if and only if the first-order approximation centered at any point $x$ underestimates $f$ everywhere.

$$f(y) \geq f(x) + (\nabla f(x))^T(y - x)$$
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- Local information $\rightarrow$ global information
- If $\nabla f(x) = 0$ then $x$ is a global minimizer of $f$
Second Order Definition

A twice differentiable $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if its derivative is nondecreasing in all directions. Formally,

$$\nabla^2 f(x) \succeq 0$$

for all $x$. 

Interpretation

Recall definition of PSD:

$z^\top \nabla^2 f(x) z > 0$ for all $z \in \mathbb{R}^n$

At $x + \delta \vec{z}$, infinitesimal change in gradient is in roughly the same direction as $\vec{z}$.

Graph of $f$ curves upwards along any line.

When $n = 1$, this is $f''(x) \geq 0$. 
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Convex Functions 3/23
The epigraph of $f$ is the set of points above the graph of $f$. Formally,

$$\text{epi}(f) = \{(x, t) : t \geq f(x)\}$$
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Epigraph Definition
$f$ is a convex function if and only if its epigraph is a convex set.
Jensen’s Inequality (General Form)

\( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is convex if and only if

- For every \( x_1, \ldots, x_k \) in the domain of \( f \), and \( \theta_1, \ldots, \theta_k \geq 0 \) such that \( \sum_i \theta_i = 1 \), we have
  \[
  f\left(\sum_i \theta_i x_i\right) \leq \sum_i \theta_i f(x_i)
  \]

- Given a probability measure \( \mathcal{D} \) on the domain of \( f \), and \( x \sim \mathcal{D} \),
  \[
  f(\mathbb{E}[x]) \leq \mathbb{E}[f(x)]
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Adding noise to $x$ can only increase $f(x)$ in expectation.
Local and Global Optimality

Local minimum

$x$ is a **local minimum** of $f$ if there is an open ball $B$ containing $x$ where $f(y) \geq f(x)$ for all $y \in B$.

Local and Global Optimality

When $f$ is convex, $x$ is a local minimum of $f$ if and only if it is a global minimum.
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Local and Global Optimality

When $f$ is convex, $x$ is a local minimum of $f$ if and only if it is a global minimum.

- This fact underlies much of the tractability of convex optimization.
Sub-level sets

Level sets of \( f(x, y) = \sqrt{x^2 + y^2} \)

The \( \alpha \)-sublevel set of \( f \) is \( \{ x \in \text{domain}(f) : f(x) \leq \alpha \} \).
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Fact

Every sub-level set of a convex function is a convex set.

- This fact also underlies tractability of convex optimization.
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- This fact also underlies tractability of convex optimization

Note: converse false, but nevertheless useful check.
Other Basic Properties

Continuity

Convex functions are continuous.
**Continuity**

Convex functions are continuous.

**Extended-value extension**

If a function $f : D \to \mathbb{R}$ is convex on its domain, and $D$ is convex, then it can be extended to a convex function on $\mathbb{R}^n$ by setting $f(x) = \infty$ whenever $x \notin D$.

This simplifies notation. Resulting function $\tilde{f} : D \to \mathbb{R} \cup \infty$ is “convex” with respect to the ordering on $\mathbb{R} \cup \infty$. 
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Functions on the reals

- **Affine:** $ax + b$
- **Exponential:** $e^{ax}$ convex for any $a \in \mathbb{R}$
- **Powers:** $x^a$ convex on $\mathbb{R}_{++}$ when $a \geq 1$ or $a \leq 0$, and concave for $0 \leq a \leq 1$
- **Logarithm:** $\log x$ concave on $\mathbb{R}_{++}$. 
Norms are convex.

\[ \|\theta x + (1 - \theta)y\| \leq \|\theta x\| + \|(1 - \theta)y\| = \theta\|x\| + (1 - \theta)\|y\| \]

- Uses both norm axioms: triangle inequality, and homogeneity.
- Applies to matrix norms, such as the spectral norm (radius of induced ellipsoid)
Norms

Norms are convex.

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- Uses both norm axioms: triangle inequality, and homogeneity.
- Applies to matrix norms, such as the spectral norm (radius of induced ellipsoid)

Max

\( \max_i x_i \) is convex

\[
\max_i (\theta x + (1 - \theta)y)_i = \max_i (\theta x_i + (1 - \theta)y_i)
\]
\[
\leq \max_i \theta x_i + \max_i (1 - \theta)y_i
\]
\[
= \theta \max_i x_i + (1 - \theta) \max_i y_i
\]

If i’m allowed to pick the maximum entry of \( \theta x \) and \( \theta y \) independently, I can do only better.
- Log-sum-exp: \( \log(e^{x_1} + e^{x_2} + \ldots + e^{x_n}) \) is convex
- Geometric mean: \( (\prod_{i=1}^{n} x_i)^{\frac{1}{n}} \) is concave
- Log-determinant: \( \log \det X \) is concave
- Quadratic form: \( x^\top A x \) is convex iff \( A \succeq 0 \)
- Other examples in book

\[
f(x, y) = \log(e^x + e^y)
\]
- Log-sum-exp: \( \log(e^{x_1} + e^{x_2} + \ldots + e^{x_n}) \) is convex
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Proving convexity often comes down to case-by-case reasoning, involving:
- Definition: restrict to line and check Jensen’s inequality
- Write down the Hessian and prove PSD
- Express as a combination of other convex functions through convexity-preserving operations (Next)
Outline

1 Convex Functions

2 Examples of Convex and Concave Functions

3 Convexity-Preserving Operations
Nonnegative Weighted Combinations

If \( f_1, f_2, \ldots, f_k \) are convex, and \( w_1, w_2, \ldots, w_k \geq 0 \), then

\[
g = w_1 f_1 + w_2 f_2 + \cdots + w_k f_k \text{ is convex.}
\]
Nonnegative Weighted Combinations

If $f_1, f_2, \ldots, f_k$ are convex, and $w_1, w_2, \ldots, w_k \geq 0$, then $g = w_1 f_1 + w_2 f_2 \ldots + w_k f_k$ is convex.

**proof ($k = 2$)**

$$g \left( \frac{x + y}{2} \right) = w_1 f_1 \left( \frac{x + y}{2} \right) + w_2 f_2 \left( \frac{x + y}{2} \right)$$

$$\leq w_1 \frac{f_1(x) + f_1(y)}{2} + w_2 \frac{f_2(x) + f_2(y)}{2}$$

$$= \frac{g(x) + g(y)}{2}$$
Nonnegative Weighted Combinations

If \( f_1, f_2, \ldots, f_k \) are convex, and \( w_1, w_2, \ldots, w_k \geq 0 \), then
\[
g = w_1 f_1 + w_2 f_2 \ldots + w_k f_k
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is convex.

Extends to integrals
\[
g(x) = \int y \, w(y) f_y(x) \text{ with } w(y) \geq 0,
\]
and therefore expectations
\[
E_y f_y(x).
\]
Nonnegative Weighted Combinations

If \( f_1, f_2, \ldots, f_k \) are convex, and \( w_1, w_2, \ldots, w_k \geq 0 \), then
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Extends to integrals
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\]
and therefore expectations \( \mathbb{E}_y f_y(x) \).

Worth Noting

Minimizing the expectation of a random convex cost function is also a convex optimization problem!

- A stochastic convex optimization problem is a convex optimization problem.
Example: Stochastic Facility Location

Average Distance

- \( k \) customers located at \( y_1, y_2, \ldots, y_k \in \mathbb{R}^n \)
- If I place a facility at \( x \in \mathbb{R}^n \), average distance to a customer is
  \[
g(x) = \sum_i \frac{1}{k} ||x - y_i||
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Example: Stochastic Facility Location

Average Distance

- $k$ customers located at $y_1, y_2, \ldots, y_k \in \mathbb{R}^n$
- If I place a facility at $x \in \mathbb{R}^n$, average distance to a customer is
  \[ g(x) = \sum_i \frac{1}{k} ||x - y_i|| \]
- Since distance to any one customer is convex in $x$, so is the average distance.
- Extends to probability measure over customers
Implication

Convex functions are a convex cone in the vector space of functions from $\mathbb{R}^n$ to $\mathbb{R}$.

The set of convex functions is the intersection of an infinite set of homogeneous linear inequalities indexed by $x, y, \theta$

$$f(\theta x + (1 - \theta)y) - \theta f(x) + (1 - \theta)f(y) \leq 0$$
Composition with Affine Function

If \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is convex, and \( A \in \mathbb{R}^{n \times m}, b \in \mathbb{R}^n \), then

\[
g(x) = f(Ax + b)
\]

is a convex function from \( \mathbb{R}^m \) to \( \mathbb{R} \).
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Proof

\((x, t) \in \text{graph}(g) \iff t = g(x) = f(Ax+b) \iff (Ax+b, t) \in \text{graph}(f)\)
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\[
(x, t) \in \text{epi}(g) \iff t \geq g(x) = f(Ax + b) \iff (Ax + b, t) \in \text{epi}(f)
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Composition with Affine Function

If $f : \mathbb{R}^n \to \mathbb{R}$ is convex, and $A \in \mathbb{R}^{n \times m}$, $b \in \mathbb{R}^n$, then

$$g(x) = f(Ax + b)$$

is a convex function from $\mathbb{R}^m$ to $\mathbb{R}$.

Proof

$$(x, t) \in \text{graph}(g) \iff t = g(x) = f(Ax+b) \iff (Ax+b, t) \in \text{graph}(f)$$

$$(x, t) \in \text{epi}(g) \iff t \geq g(x) = f(Ax + b) \iff (Ax + b, t) \in \text{epi}(f)$$

$\text{epi}(g)$ is the inverse image of $\text{epi}(f)$ under the affine mapping

$$(x, t) \mapsto (Ax + b, t)$$
Examples

- $\|Ax + b\|$ is convex
- $\max(Ax + b)$ is convex
- $\log(e^{\mathbf{a}_1^\top\mathbf{x} + b_1} + e^{\mathbf{a}_2^\top\mathbf{x} + b_2} + \ldots + e^{\mathbf{a}_n^\top\mathbf{x} + b_n})$ is convex
If $f_1, f_2$ are convex, then $g(x) = \max \{ f_1(x), f_2(x) \}$ is also convex.

Generalizes to the maximum of any number of functions, $\max_{i=1}^{k} f_i(x)$, and also to the supremum of an infinite set of functions $\sup_y f_y(x)$. 
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Generalizes to the maximum of any number of functions, $\max_{i=1}^{k} f_i(x)$, and also to the supremum of an infinite set of functions $\sup_y f_y(x)$.

$$\text{epi } g = \text{epi } f_1 \bigcap \text{epi } f_2$$
Example: Robust Facility Location

- $k$ customers located at $y_1, y_2, \ldots, y_k \in \mathbb{R}^n$
- If I place a facility at $x \in \mathbb{R}^n$, maximum distance to a customer is
  
  $$g(x) = \max_i \|x - y_i\|$$
Example: Robust Facility Location

**Maximum Distance**

- $k$ customers located at $y_1, y_2, \ldots, y_k \in \mathbb{R}^n$
- If I place a facility at $x \in \mathbb{R}^n$, maximum distance to a customer is
  $$g(x) = \max_i ||x - y_i||$$

Since distance to any one customer is convex in $x$, so is the worst-case distance.
Example: Robust Facility Location

Maximum Distance

- $k$ customers located at $y_1, y_2, \ldots, y_k \in \mathbb{R}^n$
- If I place a facility at $x \in \mathbb{R}^n$, maximum distance to a customer is $g(x) = \max_i ||x - y_i||$

Worth Noting

When a convex cost function is uncertain, minimizing the worst-case cost is also a convex optimization problem!

- A robust (in the worst-case sense) convex optimization problem is a convex optimization problem.
Other Examples

- Maximum eigenvalue of a symmetric matrix $A$ is convex in $A$

  $$\max \{ v^\top A v : \|v\| = 1 \}$$

- Sum of $k$ largest components of a vector $x$ is convex in $x$

  $$\max \left\{ \vec{1}_S \cdot x : |S| = k \right\}$$
Minimization

If $f(x, y)$ is convex and $C$ is convex and nonempty, then $g(x) = \inf_{y \in C} f(x, y)$ is convex.
Minimization

If \( f(x, y) \) is convex and \( C \) is convex and nonempty, then \( g(x) = \inf_{y \in C} f(x, y) \) is convex.

Proof (for \( C = \mathbb{R}^k \))

\( \text{epi} \ g \) is the projection of \( \text{epi} \ f \) onto hyperplane \( y = 0 \).

\[ f(x, y) = x^2 + y^2 \]

\[ g(x) = x^2 \]
Example

Distance from a convex set $C$

$$f(x, y) = \inf_{y \in C} ||x - y||$$
Composition Rules

If \( g : \mathbb{R}^n \to \mathbb{R}^k \) and \( h : \mathbb{R}^k \to \mathbb{R} \), then \( f = h \circ g \) is convex if

- \( g_i \) are convex, and \( h \) is convex and nondecreasing in each argument.
- \( g_i \) are concave, and \( h \) is convex and nonincreasing in each argument.

Proof \((n = k = 1)\)

\[
 f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)
\]
Perspective

If $f$ is convex then $g(x, t) = tf(x/t)$ is also convex.

Proof

epi $g$ is inverse image of epi $f$ under the perspective function.