Combinatorial Problems as Linear Programs

Instructor: Shaddin Dughmi
In CS, discrete problems are traditionally viewed/analyzed using discrete mathematics and combinatorics.

- Algorithms are combinatorial in nature (greedy, dynamic programming, divide and conquer, etc).

Increasingly in recent history, it is becoming clear that combining both viewpoints is the way to go.

- Better algorithms (runtime, approximation)
- Structural insights (e.g. market clearing prices in matching markets)
- Unifying theories and general results (Matroids, submodular optimization, constraint satisfaction)
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- Usually linear programs, but increasingly more general convex programs
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This is not surprising, since almost any finite family of discrete objects can be encoded as a finite subset of Euclidean space:

- Convex hull of that set is a polytope
- E.g. spanning trees, paths, cuts, TSP tours, assignments...
LP algorithms typically require representation as a “small” family of inequalities,

- Not possible in general (Say when problem is NP-hard, assuming $(P \neq NP)$)
- Shown unconditionally impossible in some cases (e.g. TSP)
Discrete Problems as Linear Programs

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- But, in many cases, polyhedra in inequality form can be shown to encode a combinatorial problems at the vertices

Next

We examine some combinatorial problems shortest path through the lense of LP and convex optimization, starting with shortest path.
Outline

1 Introduction

2 Shortest Path

3 Algorithms for Single-Source Shortest Path

4 Bipartite Matching

5 Total Unimodularity

6 Duality of Bipartite Matching and its Consequences

7 Spanning Trees

8 Flows
The Shortest Path Problem

Given a directed graph $G = (V, E)$ with cost $c_e \in \mathbb{R}$ on edge $e$, find the minimum cost path from $s$ to $t$.

- We use $n$ and $m$ to denote $|V|$ and $|E|$, respectively.
- We allow costs to be negative, but assume no negative cycles
Given a directed graph $G = (V, E)$ with cost $c_e \in \mathbb{R}$ on edge $e$, find the minimum cost path from $s$ to $t$.

- We use $n$ and $m$ to denote $|V|$ and $|E|$, respectively.
- We allow costs to be negative, but assume no negative cycles.

When costs are nonnegative, Dijkstra’s algorithm finds the shortest path from $s$ to every other node in time $O(m + n \log n)$.

Using primal/dual paradigm, we will design a polynomial-time algorithm that works when graph has negative edges but no negative cycles.
• When the graph has no negative cycles, there is a shortest path which is simple.

• When the graph has negative cycles, there may not be a shortest path from $s$ to $t$.

• In these cases, the algorithm we design can be modified to “fail gracefully” by detecting such a cycle.
  • Can be used to detect arbitrage opportunities in currency exchange networks.
When the graph has no negative cycles, there is a shortest path which is **simple**

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In the presence of negative cycles, finding the shortest **simple** path is NP-hard (by reduction from Hamiltonian cycle)
Consider the following LP

<table>
<thead>
<tr>
<th>Primal Shortest Path LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{min} \sum_{e \in E} c_e x_e )</td>
</tr>
<tr>
<td>s.t. ( \sum_{e \rightarrow v} x_e - \sum_{v \rightarrow e} x_e = \delta_v, \quad \forall v \in V. )</td>
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<tr>
<td>( x_e \geq 0, \quad \forall e \in E. )</td>
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</table>

where \( \delta_v = -1 \) if \( v = s \), \( 1 \) if \( v = t \), and 0 otherwise.
An LP Relaxation of Shortest Path

Consider the following LP

**Primal Shortest Path LP**

\[
\begin{align*}
\text{min} & \quad \sum_{e \in E} c_e x_e \\
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\end{align*}
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where \( \delta_v = -1 \) if \( v = s \), 1 if \( v = t \), and 0 otherwise.

- This is a relaxation of the shortest path problem
- Indicator vector \( x_P \) of \( s - t \) path \( P \) is a feasible solution, with cost as given by the objective
- Fractional feasible solutions may not correspond to paths
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- This is a relaxation of the shortest path problem
  - Indicator vector \( x_P \) of \( s - t \) path \( P \) is a feasible solution, with cost as given by the objective
  - Fractional feasible solutions may not correspond to paths
- A-priori, it is conceivable that optimal value of LP is less than length of shortest path.
Integrality of the Shortest Path Polyhedron

\[
\begin{align*}
\min & \quad \sum_{e \in E} c_e x_e \\
\text{s.t.} & \quad \sum_{e \to v} x_e - \sum_{v \to e} x_e = \delta_v, \quad \forall v \in V. \\
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\]

We will show that above LP encodes the shortest path problem exactly.

**Claim**

When \( c \) satisfies the no-negative-cycles property, the indicator vector of the shortest \( s - t \) path is an optimal solution to the LP.
We will use the following LP dual

**Primal LP**
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\end{align*}
\]

**Dual LP**
\[
\begin{align*}
\max & \quad y_t - y_s \\
\text{s.t.} & \quad y_v - y_u \leq c_e, \quad \forall (u, v) \in E.
\end{align*}
\]

- Interpretation of dual variables $y_v$: “height” or “potential”
- Relative potential of vertices constrained by length of edge between them (triangle inequality)
- Dual is trying to maximize relative potential of $s$ and $t$
Claim

When $c$ satisfies the no-negative-cycles property, the indicator vector of the shortest $s - t$ path is an optimal solution to the LP.
Proof Using the Dual

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- Let \( x^* \) be indicator vector of shortest \( s-t \) path
  - Feasible for primal
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Let $x^*$ be indicator vector of shortest s-t path

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Let $y^*_v$ be shortest path distance from $s$ to $v$

- Feasible for dual (by triangle inequality)
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- Let \( x^* \) be indicator vector of shortest s-t path
  - Feasible for primal
- Let \( y_v^* \) be shortest path distance from \( s \) to \( v \)
  - Feasible for dual (by triangle inequality)
- \( \sum_e c_e x_e^* = y_t^* - y_s^* \), so both \( x^* \) and \( y^* \) optimal.
A stronger statement is true:

**Integrality of Shortest Path LP**

The vertices of the polyhedral feasible region are precisely the indicator vectors of simple paths in $G$.

- Implies that there always exists an optimal solution which is a path whenever LP is bounded and feasible
- Reduces computing shortest path in graphs with no negative cycles to finding optimal vertex of LP
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**Proof**

1. LP is bounded iff $c$ satisfies no-negative-cycles
   - $\leftarrow$: previous proof
   - $\rightarrow$: If $c$ has a negative cycle, there are arbitrarily cheap “flows” along that cycle
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2. Fact: For every LP vertex $x$ there is objective $c$ such that $x$ is unique optimal. (Prove it!)
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2. Fact: For every LP vertex $x$ there is objective $c$ such that $x$ is unique optimal. (Prove it!)

3. Since such a $c$ satisfies no-negative-cycles property, our previous claim shows that $x$ is integral.
A stronger statement is true:

**Integrality of Shortest Path LP**

The vertices of the polyhedral feasible region are precisely the indicator vectors of simple paths in $G$. 

In general, the approach we took applies in many contexts: To show a polytope’s vertices integral, it suffices to show that there is an integral optimal for any objective.
1. Introduction
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4. Bipartite Matching
5. Total Unimodularity
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## Ford’s Algorithm

### Primal LP

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$$\sum_{e \rightarrow v} x_e - \sum_{v \rightarrow e} x_e = \delta_v, \quad \forall v \in V.$$  
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### Dual LP

$$\max y_t - y_s$$

s.t.

$$y_v - y_u \leq c_e, \quad \forall e = (u, v) \in E.$$  

For convenience, add $$(s, v)$$ of length $\infty$ when one doesn’t exist.

### Ford’s Algorithm

1. \(y_v = c_{(s, v)}\) and \(\text{pred}(v) \leftarrow s\) for \(v \neq s\)
2. \(y_s \leftarrow 0, \text{pred}(s) = \text{null}\).
3. While some dual constraint is violated, i.e. \(y_v > y_u + c_e\) for some \(e = (u, v)\)
   - \(y_v \leftarrow y_u + c_e\)
   - Set \(\text{pred}(v) = u\)
4. Output the path \(t, \text{pred}(t), \text{pred}(\text{pred}(t)), \ldots, s\).
## Correctness

### Lemma (Loop Invariant 1)

Assuming no negative cycles, \( \text{pred} \) defines a path \( P \) from \( s \) to \( t \), of length at most \( y_t - y_s \).

### Interpretation

- Ford’s algorithm maintains an (initially infeasible) dual \( y \)
- Also maintains feasible primal \( P \) of length \( \leq \) dual objective \( y_t - y_s \)
- Iteratively “fixes” dual \( y \), tending towards feasibility
- Once \( y \) is feasible, weak duality implies \( P \) optimal.
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Correctness follows from loop invariant 1 and termination condition.

**Theorem (Correctness)**

If Ford’s algorithm terminates, then it outputs a shortest path from \(s\) to \(t\).
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Theorem (Correctness)

If Ford’s algorithm terminates, then it outputs a shortest path from \(s\) to \(t\).

Algorithms of this form, that output a matching primal and dual solution, are called Primal-Dual Algorithms.
Termination

Lemma (Loop Invariant 2)

Assuming no negative cycles, \( y_v \) is the length of some simple path from \( s \) to \( v \).
Termination

Lemma (Loop Invariant 2)
Assuming no negative cycles, $y_v$ is the length of some simple path from $s$ to $v$.

Theorem (Termination)
When the graph has no negative cycles, Ford’s algorithm terminates in a finite number of steps.

Proof
- The graph has a finite number $N$ of simple paths.
- By loop invariant 2, every dual variable $y_v$ is the length of some simple path.
- Dual variables are nonincreasing throughout algorithm, and one decreases each iteration.
- There can be at most $nN$ iterations.
Observation: Single sink shortest paths

**Ford’s Algorithm**

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3. While some dual constraint is violated, i.e. \( y_v > y_u + c_e \) for some \( e = (u, v) \)
   - \( y_v \leftarrow y_u + c_e \)
   - Set \( \text{pred}(v) = u \)
4. Output the path \( t, \text{pred}(t), \text{pred}(\text{pred}(t)), \ldots, s \).

**Observation**

Algorithm does not depend on \( t \) till very last step. So essentially solves the single-source shortest path problem. i.e. finds shortest paths from \( s \) to all other vertices \( v \).
We prove Loop Invariant 1 through two Lemmas

**Lemma (Loop Invariant 1a)**
For every node $w$, we have $y_w - y_{pred(w)} \geq c_{pred(w),w}$

**Proof**
- **Fix $w$**
- **Holds at first iteration**
- **Preserved by Induction on iterations**
  - If neither $y_w$ nor $y_{pred(w)}$ updated, nothing changes.
  - If $y_w$ (and $pred(w)$) updated, then $y_w \leftarrow y_{pred(w)} + c_{pred(w),w}$
  - $y_{pred(w)}$ updated, it only goes down, preserving inequality.
Loop Invariant 1

Lemma (Invariant 1b)

Assuming no negative cycles, \text{pred} forms a directed tree rooted out of \text{s}.

We denote this path from \text{s} to a node \text{w} by \text{P}(s, w).

Proof

- Holds at first iteration
- For a contradiction, consider iteration of first violation
  - \text{v} and \text{u} with \( y_v > y_u + c_{u,v} \)
  - \text{P}(s, u) passes through \text{v}
    - Otherwise tree property preserved by \text{pred}(v) \leftarrow \text{u}
- Let \text{P}(v, u) be the portion of \text{P}(s, u) starting at \text{v}.
- By Invariant 1a, and telescoping sum, length of \text{P}(v, u) is at most \( y_u - y_v \).
- Length of cycle \{\text{P}(v, u), (u, v)\} at most \( y_u - y_v + c_{u,v} < 0 \).
Summarizing Loop Invariant 1

Lemma (Invariant 1a)
For every node $w$, we have $y_w - y_{\text{pred}(w)} \geq c_{\text{pred}(w),w}$.

- By telescoping sum, can bound $y_w - y_s$ when pred leads back to $s$

Lemma (Invariant 1b)
Assuming no negative cycles, pred forms a directed tree rooted out of $s$.

- Implies that $y_s$ remains 0

Corollary (Loop Invariant 1)
Assuming no negative cycles, $\text{pred}$ defines a path $P(s,w)$ from $s$ to each node $w$, of length at most $y_w - y_s = y_w$. 

Algorithms for Single-Source Shortest Path
Lemma (Loop Invariant 2)

Assuming no negative cycles, $y_w$ is the length of some simple path $Q(s, w)$ from $s$ to $w$, for all $w$.

Proof is technical, by induction, so we will skip. Instead, we will modify Ford's algorithm to guarantee polynomial time termination.
Bellman-Ford Algorithm

The following algorithm fixes an (arbitrary) order on edges $E$

1. $y_v = c(s,v)$ and $\text{pred}(v) \leftarrow s$ for $v \neq s$
2. $y_s \leftarrow 0$, $\text{pred}(s) = \text{null}$.
3. While $y$ is infeasible for the dual
   - For $e = (u,v)$ in order, if $y_v > y_u + c_e$ then
     - $y_v \leftarrow y_u + c_e$
     - Set $\text{pred}(v) = u$
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4. Output the path $t, \text{pred}(t), \text{pred}(\text{pred}(t)), \ldots, s$.

Note

Correctness follows from the correctness of Ford’s Algorithm.
**Theorem**

*Bellman-Ford terminates after $n - 1$ scans through $E$, for a total runtime of $O(nm)$.***
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Follows immediately from the following Lemma

**Lemma**

After $k$ scans through $E$, vertices $v$ with a shortest $s - v$ path consisting of $\leq k$ edges are correctly labeled. (i.e., $y_v = \text{distance}(s, v)$)
Proof

Lemma

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Proof

- Holds for $k = 0$
- By induction on $k$.
  - Assume it holds for $k - 1$.
  - Let $v$ be a node with a shortest path $P$ from $s$ with $k$ edges.
  - $P = \{Q, e\}$, for some $e = (u, v)$ and $s - u$ path $Q$, where $Q$ is a shortest $s - u$ path and $Q$ has $k - 1$ edges.
  - By inductive hypothesis, $u$ is correctly labeled just before $e$ is scanned – i.e. $y_u = \text{distance}(s, u)$.
  - Therefore, $v$ is correctly labeled $y_v \leftarrow y_u + c_{u,v} = \text{distance}(s, v)$ after $e$ is scanned
A Note on Negative Cycles

Question
What if there are negative cycles? What does that say about LP? What about Ford’s algorithm?
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5. Total Unimodularity
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The Max-Weight Bipartite Matching Problem

Given a bipartite graph $G = (V, E)$, with $V = L \cup R$, and weights $w_e$ on edges $e$, find a maximum weight matching.

- **Matching**: a set of edges covering each node at most once
- We use $n$ and $m$ to denote $|V|$ and $|E|$, respectively.
- Equivalent to maximum weight / minimum cost perfect matching.
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Our focus will be less on algorithms, and more on using polyhedral interpretation to gain insights about a combinatorial problem.
The Bipartite Matching LP is given by:

\[
\begin{align*}
\text{max} \quad & \sum_{e \in E} w_e x_e \\
\text{s.t.} \quad & \sum_{e \in \delta(v)} x_e \leq 1, \quad \forall v \in V. \\
\quad & x_e \geq 0, \quad \forall e \in E.
\end{align*}
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An LP Relaxation of Bipartite Matching

Bipartite Matching LP

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\end{align*}
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- Feasible region is a polytope \( \mathcal{P} \) (i.e. a bounded polyhedron)
- This is a relaxation of the bipartite matching problem
  - Integer points in \( \mathcal{P} \) are the indicator vectors of matchings.

\[
\mathcal{P} \cap \mathbb{Z}^m = \{x_M : M \text{ is a matching}\}
\]
The feasible region of the matching LP is the convex hull of indicator vectors of matchings.

\[ \mathcal{P} = \text{convexhull} \{ x_M : M \text{ is a matching} \} \]
The integrality of the Bipartite Matching Polytope can be formulated as follows:

\[
\sum_{e \in \delta(v)} x_e \leq 1, \quad \forall v \in V.
\]
\[
x_e \geq 0, \quad \forall e \in E.
\]

**Theorem**

The feasible region of the matching LP is the convex hull of indicator vectors of matchings.

\[
P = \text{convexhull } \{x_M : M \text{ is a matching}\}
\]

**Note**

- This is the strongest guarantee you could hope for of an LP relaxation of a combinatorial problem.
- Solving LP is equivalent to solving the combinatorial problem.
- Stronger guarantee than shortest path LP from last time.
Suffices to show that all vertices are integral (why?)
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Consider $x \in P$ non-integral, we will show that $x$ is not a vertex.
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Consider $x \in P$ non-integral, we will show that $x$ is not a vertex.
Let $H$ be the subgraph formed by edges with $x_e \in (0, 1)$
Suffices to show that all vertices are integral (why?)

Consider $x \in P$ non-integral, we will show that $x$ is not a vertex.

Let $H$ be the subgraph formed by edges with $x_e \in (0, 1)$

$H$ either contains a cycle, or else a maximal path which is simple.
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Consider $x \in \mathcal{P}$ non-integral, we will show that $x$ is not a vertex.
Let $H$ be the subgraph formed by edges with $x_e \in (0, 1)$
$H$ either contains a cycle, or else a maximal path which is simple.
Case 1: Cycle $C$

- Let $C = (e_1, \ldots, e_k)$, with $k$ even
- There is $\epsilon > 0$ such that adding $\pm \epsilon (+1, -1, \ldots, +1, -1)$ to $x_C$ preserves feasibility
- $x$ is the midpoint of $x + \epsilon (+1, -1, \ldots, +1, -1)_C$ and $x - \epsilon (+1, -1, \ldots, +1, -1)_C$, so $x$ is not a vertex.
Case 2: Maximal Path $P$

- Let $P = (e_1, \ldots, e_k)$, going through vertices $v_0, v_1, \ldots, v_k$.
- By maximality, $e_1$ is the only edge of $v_0$ with non-zero $x$-weight.
  - Similarly for $e_k$ and $v_k$.
- There is $\epsilon > 0$ such that adding $\pm \epsilon (+1, -1, \ldots, \pm 1)$ to $x_P$ preserves feasibility.
- $x$ is the midpoint of $x + \epsilon (+1, -1, \ldots, \pm 1)_P$ and $x - \epsilon (+1, -1, \ldots, \pm 1)_P$, so $x$ is not a vertex.
Related Fact: Birkhoff Von-Neumann Theorem

\[
\sum_{e \in \delta(v)} x_e = 1, \quad \forall v \in V.
\]
\[
x_e \geq 0, \quad \forall e \in E.
\]

- The analogous statement holds for the perfect matching LP above, by an essentially identical proof.
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- When bipartite graph is complete and has the same # of nodes on either side, can be equivalently phrased as a property of matrices.
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**Birkhoff Von-Neumann Theorem**

The set of $n \times n$ doubly stochastic matrices is the convex hull of $n \times n$ permutation matrices.

\[
\begin{pmatrix}
0.5 & 0.5 \\
0.5 & 0.5
\end{pmatrix} = 0.5 \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} + 0.5 \begin{pmatrix}
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\end{pmatrix}
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By Caratheodory’s theorem, we can express every doubly stochastic matrix as a convex combination of \( n^2 + 1 \) permutation matrices.

We will see later: this decomposition can be computed efficiently!
Total Unimodularity

We could have proved integrality of the bipartite matching LP using a more general tool.

**Definition**

A matrix $A$ is **Totally Unimodular** if every square submatrix has determinant 0, +1 or −1.

**Theorem**

If $A \in \mathbb{R}^{m \times n}$ is totally unimodular, and $b$ is an integer vector, then \[ \{ x : Ax \leq b, x \geq 0 \} \] has integer vertices.
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*If* $A \in \mathbb{R}^{m \times n}$ *is totally unimodular, and* $b$ *is an integer vector, then* \{ $x : Ax \leq b, x \geq 0$ \} *has integer vertices.*

**Proof**

- Non-zero entries of vertex $x$ are solution of $A'x' = b'$ for some nonsignular square submatrix $A'$ and corresponding sub-vector $b'$.  
- Cramer’s rule:  

$$x_i' = \frac{\det(A_i'|b')}{{\det}A'}$$
Claim

The constraint matrix of the bipartite matching LP is totally unimodular.
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Proof
- $A_{ve} = 1$ if $e$ incident on $v$, and 0 otherwise.
- By induction on size of submatrix $A'$. Trivial for base case $k = 1$. 

$$\sum_{e \in \delta(v)} x_e \leq 1, \quad \forall v \in V.$$
Total Unimodularity of Bipartite Matching

\[ \sum_{e \in \delta(v)} x_e \leq 1, \quad \forall v \in V. \]

**Claim**

The constraint matrix of the bipartite matching LP is totally unimodular.

**Proof**

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- If \( A' \) has all-zero column, then \( \det A' = 0 \)
The constraint matrix of the bipartite matching LP is totally unimodular.

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- If \( A' \) has column with single 1, then holds by induction.
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- \( A_{ve} = 1 \) if \( e \) incident on \( v \), and 0 otherwise.
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- If \( A' \) has all-zero column, then \( \det A' = 0 \)
- If \( A' \) has column with single 1, then holds by induction.
- If all columns of \( A' \) have two 1's,
  - Partition rows (vertices) into \( L \) and \( R \)
  - Sum of rows \( L \) is \((1, 1, \ldots, 1)\), similarly for \( R \)
  - \( A' \) is singular, so \( \det A' = 0 \).
Outline

1. Introduction
2. Shortest Path
4. Bipartite Matching
5. Total Unimodularity
6. Duality of Bipartite Matching and its Consequences
7. Spanning Trees
8. Flows
Primal and Dual LPs

**Primal LP**

\[
\begin{align*}
\text{max } & \sum_{e \in E} w_e x_e \\
\text{s.t. } & \sum_{e \in \delta(v)} x_e \leq 1, \quad \forall v \in V. \\
& x_e \geq 0, \quad \forall e \in E.
\end{align*}
\]

**Dual LP**

\[
\begin{align*}
\text{min } & \sum_{v \in V} y_v \\
\text{s.t. } & y_u + y_v \geq w_e, \quad \forall e = (u, v) \in E. \\
& y_v \geq 0, \quad \forall v \in V.
\end{align*}
\]

- Primal interpretation: Player 1 looking to build a set of projects
  - Each edge \( e \) is a project generating “profit” \( w_e \)
  - Each project \( e = (u, v) \) needs two resources, \( u \) and \( v \)
  - Each resource can be used by at most one project at a time
  - Must choose a profit-maximizing set of projects
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  - Each resource can be used by at most one project at a time
  - Must choose a profit-maximizing set of projects

- **Dual interpretation:** Player 2 looking to buy resources
  - Offer a price \( y_v \) for each resource.
  - Prices should incentivize player 1 to sell resources
  - Want to pay as little as possible.
Vertex Cover Interpretation

**Primal LP**

\[
\text{max } \sum_{e \in E} x_e \\
\text{s.t. } \sum_{e \in \delta(v)} x_e \leq 1, \quad \forall v \in V. \\
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When edge weights are 1, binary solutions to dual are vertex covers

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\]

**Definition**

\( C \subseteq V \) is a vertex cover if every \( e \in E \) has at least one endpoint in \( C \).
Vertex Cover Interpretation

**Primal LP**

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\begin{align*}
\text{max} & \quad \sum_{e \in E} x_e \\
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\]

When edge weights are 1, binary solutions to dual are vertex covers.

**Definition**

*C* \(\subseteq V\) is a **vertex cover** if every \(e \in E\) has at least one endpoint in *C*.

*Dual is a relaxation of the minimum vertex cover problem for bipartite graphs.*

*By weak duality: min-vertex-cover \(\geq\) max-cardinality-matching*
König’s Theorem

**Primal LP**

\[
\max \sum_{e \in E} x_e \\
\text{s.t.} \\
\sum_{e \in \delta(v)} x_e \leq 1, \quad \forall v \in V. \\
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y_u + y_v \geq 1, \quad \forall e = (u, v) \in E. \\
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\]

König’s Theorem

In a bipartite graph, the cardinality of the maximum matching is equal to the cardinality of the minimum vertex cover.

i.e. the dual LP has an optimal integral solution

---

Duality of Bipartite Matching and its Consequences
Let \( M(G) \) be a max cardinality of a matching in \( G \)
Let \( C(G) \) be min cardinality of a vertex cover in \( G \)
We already proved that \( M(G) \leq C(G) \)
We will prove \( C(G) \leq M(G) \) by induction on number of nodes in \( G \).
Let \( y \) be an optimal dual, and \( v \) a vertex with \( y_v > 0 \).
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By complementary slackness, every maximum cardinality matching must match $v$. 

Note: Could have proved the same using total unimodularity.
Let $y$ be an optimal dual, and $v$ a vertex with $y_v > 0$.

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- $M(G \setminus v) = M(G) - 1$
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By inductive hypothesis, \( C(G \setminus v) = M(G \setminus v) = M(G) - 1 \)

\[ C(G) \leq C(G \setminus v) + 1 = M(G). \]
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Vertex covers can serve as a certificate of optimality for bipartite matchings, and vice versa.
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Like maximum cardinality matching, minimum vertex cover in bipartite graphs can be formulated as an LP, and solved in polynomial time.
Consequences of König’s Theorem

- Vertex covers can serve as a certificate of optimality for bipartite matchings, and vice versa.
- Like maximum cardinality matching, minimum vertex cover in bipartite graphs can be formulated as an LP, and solved in polynomial time.
- The same is true for the maximum independent set problem in bipartite graphs.
  - $C'$ is a vertex cover iff $V \setminus C'$ is an independent set.
The Minimum Cost Spanning Tree Problem

Given a connected undirected graph $G = (V, E)$, and costs $c_e$ on edges $e$, find a minimum cost spanning tree of $G$.

- **Spanning Tree**: an acyclic set of edges connecting every pair of nodes
- When graph is disconnected, can search for min-cost spanning forest instead
- We use $n$ and $m$ to denote $|V|$ and $|E|$, respectively.
Kruskal’s Algorithm

The minimum spanning tree problem can be solved efficiently by a simple greedy algorithm

Kruskal’s algorithm

1. $T \leftarrow \emptyset$
2. Sort edges in increasing order of cost
3. For each edge $e$ in order
   - if $T \cup e$ is acyclic, add $e$ to $T$. 

Proof of correctness is via a simple exchange argument. Generalizes to Matroids
Kruskal’s Algorithm

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- Proof of correctness is via a simple exchange argument.
- Generalizes to **Matroids**

Spanning Trees
MST Linear Program

**MST LP**

minimize \[ \sum_{e \in E} c_e x_e \]

subject to \[ \sum_{e \in E} x_e = n - 1 \]
\[ \sum_{e \subseteq X} x_e \leq |X| - 1, \quad \text{for } X \subset V. \]
\[ x_e \geq 0, \quad \text{for } e \in E. \]

Theorem: The feasible region of the above LP is the convex hull of spanning trees.

Proof by finding a dual solution with cost matching the output of Kruskal’s algorithm.

Generalizes to Matroids

Note: this LP has an exponential (in \( n \)) number of constraints.
MST LP

\[ \begin{align*}
& \text{minimize} & & \sum_{e \in E} c_e x_e \\
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- Note: this LP has an exponential (in $n$) number of constraints
A **separation oracle** for a linear program with feasible set \( \mathcal{P} \subseteq \mathbb{R}^m \) is an algorithm which takes as input \( x \in \mathbb{R}^m \), and either certifies that \( x \in \mathcal{P} \) or identifies a violated constraint.
Solving the MST Linear Program

Definition

A separation oracle for a linear program with feasible set $\mathcal{P} \subseteq \mathbb{R}^m$ is an algorithm which takes as input $x \in \mathbb{R}^m$, and either certifies that $x \in \mathcal{P}$ or identifies a violated constraint.

Theorem

A linear program with a polynomial number of variables is solvable in polynomial time if and only if it admits a polynomial time separation oracle.

Follows from the ellipsoid method, which we will see next week.
Solving the MST Linear Program

### Primal LP

**minimize** \( \sum_{e \in E} c_e x_e \)

**subject to**

\[ \sum_{e \subseteq X} x_e \leq |X| - 1, \quad \text{for } X \subset V. \]

\[ \sum_{e \in E} x_e = n - 1 \]

\[ x_e \geq 0, \quad \text{for } e \in E. \]

- **Given** \( x \in \mathbb{R}^m \), separation oracle must find a violated constraint if one exists
Solving the MST Linear Program

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---

- **Given** \( x \in \mathbb{R}^m \), separation oracle must find a violated constraint if one exists.
- **Reduces to finding** \( X \subset V \) with \( \sum_{e \subseteq X} x_e > |X| - 1 \), if one exists.
  - Equivalently \( \frac{1+\sum_{e \subseteq X} x_e}{|X|} > 1 \).
Solving the MST Linear Program

Primal LP

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\text{minimize} & \quad \sum_{e \in E} c_e x_e \\
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- In turn, this reduces to maximizing \( \frac{1 + \sum_{e \subseteq X} x_e}{|X|} \) over \( X \)
Solving the MST Linear Program

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We will see how to do this efficiently later in the class, since \( \frac{1 + \sum_{e \subseteq X} x_e}{|X|} \) is a supermodular function of the set \( X \).
The LP formulation of spanning trees has many applications. We will look at one contrived yet simple application that shows the flexibility enabled by polyhedral formulation.

**Fault-Tolerant MST**

- Your tree is an overlay network on the internet used to transmit data.
- A hacker is looking to attack your tree, by knocking off one of the edges of the graph.
- You can foil the hacker by choosing a random tree.
- The hacker knows the algorithm you use, but not your random coins.
Fault-tolerant MST LP

minimize \[ \sum_{e \in E} c_e x_e \]
subject to \[ \sum_{e \subseteq X} x_e \leq |X| - 1, \quad \text{for } X \subseteq V. \]
\[ \sum_{e \in E} x_e = n - 1 \]
\[ x_e \leq p, \quad \text{for } e \in E. \]
\[ x_e \geq 0, \quad \text{for } e \in E. \]

- Above LP can be solved efficiently
- Can interpret resulting fractional spanning tree \( x \) as a recipe for a probability distribution over trees \( T \)
  - \( e \in T \) with probability \( x_e \)
  - Since \( x_e \leq p \), no edge is in the tree with probability more than \( p \).
Fault-tolerant MST LP

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\[ x_e \leq p, \quad \text{for } e \in E. \]
\[ x_e \geq 0, \quad \text{for } e \in E. \]

- Such a probability distribution exists!
Fault-tolerant MST LP

minimize $\sum_{e \in E} c_e x_e$

subject to

$\sum_{e \subseteq X} x_e \leq |X| - 1$, for $X \subset V$.

$\sum_{e \in E} x_e = n - 1$

$x_e \leq p$, for $e \in E$.

$x_e \geq 0$, for $e \in E$.

Such a probability distribution exists!

- $x$ is in the (original) MST polytope
- Caratheodory’s theorem: $x$ is a convex combination of $m + 1$ vertices of MST polytope
- By integrality of MST polytope: $x$ is the “expectation” of a probability distribution over spanning trees.
Fault-tolerant MST LP

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subject to
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Consequence of Ellipsoid algorithm: can compute such a decomposition of $x$ efficiently!
Outline

1. Introduction
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4. Bipartite Matching
5. Total Unimodularity
6. Duality of Bipartite Matching and its Consequences
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The Maximum Flow Problem

Given a directed graph \( G = (V, E) \) with capacities \( u_e \) on edges \( e \), a source node \( s \), and a sink node \( t \), find a maximum flow from \( s \) to \( t \) respecting the capacities.

\[
\text{maximize } \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e \\
\text{subject to } \sum_{e \in \delta^-(v)} x_e = \sum_{e \in \delta^+(v)} x_e, \quad \text{for } v \in V \setminus \{s, t\}. \\
x_e \leq u_e, \quad \text{for } e \in E. \\
x_e \geq 0, \quad \text{for } e \in E.
\]

Can be computed either by solving the LP, or by a combinatorial algorithm such as Ford Fulkerson.
Primal LP

$$\max \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e$$

s.t.
$$\sum_{e \in \delta^-(v)} x_e = \sum_{e \in \delta^+(v)} x_e, \quad \forall v \in V \setminus \{s, t\}$$
$$x_e \leq u_e, \quad \forall e \in E.$$  
$$x_e \geq 0, \quad \forall e \in E.$$  

Dual LP (Simplified)

$$\min \sum_{e \in E} u_e z_e$$

s.t.
$$y_v - y_u \leq z_e, \quad \forall e = (u, v) \in E.$$  
$$y_s = 0$$  
$$y_t = 1$$  
$$z_e \geq 0, \quad \forall e \in E.$$  

- Dual solution describes fraction $z_e$ of each edge to fractionally cut.
Primal LP

\[
\text{max } \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e
\]

s.t.
\[
\sum_{e \in \delta^-(v)} x_e = \sum_{e \in \delta^+(v)} x_e, \quad \forall v \in V \setminus \{s, t\}
\]
\[
x_e \leq u_e, \quad \forall e \in E.
\]
\[
x_e \geq 0, \quad \forall e \in E.
\]

Dual LP (Simplified)

\[
\text{min } \sum_{e \in E} u_e z_e
\]

s.t.
\[
y_v - y_u \leq z_e, \quad \forall e = (u, v) \in E.
\]
\[
y_s = 0
\]
\[
y_t = 1
\]
\[
z_e \geq 0, \quad \forall e \in E.
\]

- Dual solution describes fraction \(z_e\) of each edge to fractionally cut
- Dual constraints require that at least 1 edge is cut on every path from \(s\) to \(t\).

\[
\sum_{(u, v) \in P} z_{uv} \geq \sum_{(u, v) \in P} y_v - y_u = y_t - y_s = 1
\]
Every integral $s - t$ cut is feasible.
**Primal LP**

\[
\begin{align*}
\text{max} & \quad \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e \\
\text{s.t.} & \quad \sum_{e \in \delta^-(v)} x_e = \sum_{e \in \delta^+(v)} x_e, \quad \forall v \in V \setminus \{s, t\} \\
& \quad x_e \leq u_e, \quad \forall e \in E. \\
& \quad x_e \geq 0, \quad \forall e \in E.
\end{align*}
\]

**Dual LP (Simplified)**

\[
\begin{align*}
\text{min} & \quad \sum_{e \in E} u_e z_e \\
\text{s.t.} & \quad y_v - y_u \leq z_e, \quad \forall e = (u, v) \in E. \\
& \quad y_s = 0 \\
& \quad y_t = 1 \\
& \quad z_e \geq 0, \quad \forall e \in E.
\end{align*}
\]

- Every integral \( s - t \) cut is feasible.
- By weak duality: max flow \( \leq \) minimum cut
Every integral $s - t$ cut is feasible.

By weak duality: max flow $\leq$ minimum cut

Ford-Fulkerson shows that max flow $=$ min cut

i.e. dual has integer optimal
Primal LP

\[
\begin{align*}
\text{max} & \quad \sum_{e \in \delta^+ (s)} x_e - \sum_{e \in \delta^- (s)} x_e \\
\text{s.t.} & \quad \sum_{e \in \delta^- (v)} x_e = \sum_{e \in \delta^+ (v)} x_e, \quad \forall v \in V \setminus \{s, t\} \\
& \quad x_e \leq u_e, \quad \forall e \in E. \\
& \quad x_e \geq 0, \quad \forall e \in E.
\end{align*}
\]

Dual LP (Simplified)

\[
\begin{align*}
\text{min} & \quad \sum_{e \in E} u_e z_e \\
\text{s.t.} & \quad y_v - y_u \leq z_e, \quad \forall e = (u, v) \in E. \\
& \quad y_s = 0 \\
& \quad y_t = 1 \\
& \quad z_e \geq 0, \quad \forall e \in E.
\end{align*}
\]

Every integral \( s - t \) cut is feasible.

By weak duality: \( \text{max flow} \leq \text{minimum cut} \)

Ford-Fulkerson shows that \( \text{max flow} = \text{min cut} \)

- i.e. dual has integer optimal

Ford-Fulkerson also shows that there is an integral optimal flow when capacities are integer.
Generalizations of Max Flow

$$\max \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e$$

s.t.

$$\sum_{e \in \delta^-(v)} x_e = \sum_{e \in \delta^+(v)} x_e, \quad \forall v \in V \setminus \{s, t\}.$$ 

$$x_e \leq u_e, \quad \forall e \in E.$$ 

$$x_e \geq 0, \quad \forall e \in E.$$ 

Writing as an LP shows that many generalizations are also tractable.
Generalizations of Max Flow

\[
\max \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e
\]

s.t.
\[
\sum_{e \in \delta^-(v)} x_e = \sum_{e \in \delta^+(v)} x_e, \quad \forall v \in V \setminus \{s, t\}.
\]
\[
x_e \leq u_e, \quad \forall e \in E.
\]
\[
x_e \geq 0, \quad \forall e \in E.
\]

Writing as an LP shows that many generalizations are also tractable

- Lower and upper bound constraints on flow: \( \ell_e \leq x_e \leq u_e \)
Generalizations of Max Flow

\[
\begin{align*}
\text{max} & \quad \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e \\
\text{s.t.} & \quad \sum_{e \in \delta^-(v)} x_e = \sum_{e \in \delta^+(v)} x_e, \quad \forall v \in V \setminus \{s, t\}.
\end{align*}
\]

\[
x_e \leq u_e, \quad \forall e \in E.
\]

\[
x_e \geq 0, \quad \forall e \in E.
\]

Writing as an LP shows that many generalizations are also tractable

- Lower and upper bound constraints on flow: \( \ell_e \leq x_e \leq u_e \)
- Minimum cost flow of a certain amount \( r \)
  - Objective \( \min \sum_e c_e x_e \)
  - Additional constraint: \( \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e = r \)
Generalizations of Max Flow

\[
\begin{align*}
\text{max} & \quad \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e \\
\text{s.t.} & \quad \sum_{e \in \delta^-(v)} x_e = \sum_{e \in \delta^+(v)} x_e, \quad \forall v \in V \setminus \{s, t\}.
\end{align*}
\]

\[x_e \leq u_e, \quad \forall e \in E.\]

\[x_e \geq 0, \quad \forall e \in E.\]

Writing as an LP shows that many generalizations are also tractable:

- Lower and upper bound constraints on flow: \(l_e \leq x_e \leq u_e\)
- Minimum cost flow of a certain amount \(r\)
  - Objective \(\min \sum_e c_e x_e\)
  - Additional constraint: \(\sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e = r\)
- Multiple commodities sharing the network
Generalizations of Max Flow

\[
\max \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e
\]

s.t.
\[
\sum_{e \in \delta^-(v)} x_e = \sum_{e \in \delta^+(v)} x_e, \quad \forall v \in V \setminus \{s, t\}.
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Writing as an LP shows that many generalizations are also tractable

- Lower and upper bound constraints on flow: \( l_e \leq x_e \leq u_e \)
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  - Objective \( \min \sum_e c_e x_e \)
  - Additional constraint: \( \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e = r \)
- Multiple commodities sharing the network
- . . .
Minimum Congestion Flow

You are given a directed graph $G = (V, E)$ with congestion functions $c_e(.)$ on edges $e$, a source node $s$, a sink node $t$, and a desired flow amount $r$. Find a minimum average congestion flow from $s$ to $t$.

\[
\begin{align*}
\text{minimize} & \quad \sum_e x_e c_e(x_e) \\
\text{subject to} & \quad \sum_{e \in \delta^+(s)} x_e - \sum_{e \in \delta^-(s)} x_e = r \\
& \quad \sum_{e \in \delta^-(v)} x_e = \sum_{e \in \delta^+(v)} x_e, \quad \text{for } v \in V \setminus \{s, t\}. \\
& \quad x_e \geq 0, \quad \text{for } e \in E. 
\end{align*}
\]

When $c_e(.)$ are polynomials with nonnegative co-efficients, e.g. $c_e(x) = a_e x^2 + b_e x + c_e$ with $a_e, b_e, c_e \geq 0$, this is a (non-linear) convex program.