<table>
<thead>
<tr>
<th>#</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Linear Programming Basics</td>
</tr>
<tr>
<td>2</td>
<td>Duality and Its Interpretations</td>
</tr>
<tr>
<td>3</td>
<td>Properties of Duals</td>
</tr>
<tr>
<td>4</td>
<td>Weak and Strong Duality</td>
</tr>
<tr>
<td>5</td>
<td>Consequences of Duality</td>
</tr>
<tr>
<td>6</td>
<td>Uses and Examples of Duality</td>
</tr>
<tr>
<td>7</td>
<td>Solvability of LP</td>
</tr>
</tbody>
</table>
Outline

1. Linear Programming Basics
2. Duality and Its Interpretations
3. Properties of Duals
4. Weak and Strong Duality
5. Consequences of Duality
6. Uses and Examples of Duality
7. Solvability of LP
The forefather of convex optimization problems, and the most ubiquitous.

Developed by Kantorovich during World War II (1939) for planning the Soviet army’s expenditures and returns. Kept secret.

Discovered a few years later by George Dantzig, who in 1947 developed the simplex method for solving linear programs.

John von Neumann developed LP duality in 1947, and applied it to game theory.

Polynomial-time algorithms: Ellipsoid method (Khachiyan 1979), interior point methods (Karmarkar 1984).
minimize (or maximize) \hfill c^\top x
subject to
\begin{align*}
a_i^\top x &\leq b_i, \quad \text{for } i \in C^1. \\
a_i^\top x &\geq b_i, \quad \text{for } i \in C^2. \\
a_i^\top x &= b_i, \quad \text{for } i \in C^3.
\end{align*}

- **Decision variables:** \( x \in \mathbb{R}^n \)
- **Parameters:**
  - \( c \in \mathbb{R}^n \) defines the linear objective function
  - \( a_i \in \mathbb{R}^n \) and \( b_i \in \mathbb{R} \) define the \( i \)'th constraint.
maximize $c^T x$
subject to $a_i^T x \leq b_i$, for $i = 1, \ldots, m$.

$x_j \geq 0$, for $j = 1, \ldots, n$.

Every LP can be transformed to this form

- minimizing $c^T x$ is equivalent to maximizing $-c^T x$
- $\geq$ constraints can be flipped by multiplying by $-1$
- Each equality constraint can be replaced by two inequalities
- Uconstrained variable $x_j$ can be replaced by $x_j^+ - x_j^-$, where both $x_j^+$ and $x_j^-$ are constrained to be nonnegative.
A 2-D example

maximize \quad x_1 + x_2
subject to \quad x_1 + 2x_2 \leq 2
\quad 2x_1 + x_2 \leq 2
\quad x_1, x_2 \geq 0
Application: Optimal Production

- $n$ products, $m$ raw materials
- Every unit of product $j$ uses $a_{ij}$ units of raw material $i$
- There are $b_i$ units of material $i$ available
- Product $j$ yields profit $c_j$ per unit
- Facility wants to maximize profit subject to available raw materials

$$\begin{align*}
\text{maximize} & \quad c^T x \\
\text{subject to} & \quad a_i^T x \leq b_i, \quad \text{for } i = 1, \ldots, m. \\
& \quad x_j \geq 0, \quad \text{for } j = 1, \ldots, n.
\end{align*}$$
**Terminology**

- **Hyperplane**: The region defined by a linear equality $a_i^T x \leq b_i$.
- **Halfspace**: The region defined by a linear inequality $a_i^T x \leq b_i$.
- **Polyhedron**: The intersection of a set of linear inequalities.
  - Feasible region of an LP is a polyhedron.
- **Polytope**: Bounded polyhedron.
  - Equivalently: convex hull of a finite set of points.
- **Vertex**: A point $x$ is a vertex of polyhedron $P$ if $\exists y \neq 0$ with $x + y \in P$ and $x - y \in P$.
- **Face of $P$**: The intersection with $P$ of a hyperplane $H$ disjoint from the interior of $P$. 

![Diagram of hyperplanes and polyhedron](image)
Basic Facts about LPs and Polyhedrons

Fact
Feasible regions of LPs (i.e. polyhedrons) are convex
Basic Facts about LPs and Polyhedrons

Fact

Feasible regions of LPs (i.e. polyhedrons) are convex

Fact

Set of optimal solutions of an LP is convex

- In fact, a face of the polyhedron
- intersection of $P$ with hyperplane $c^\top x = OPT$
Basic Facts about LPs and Polyhedrons

Fact
Feasible regions of LPs (i.e. polyhedrons) are convex

Fact
Set of optimal solutions of an LP is convex
- In fact, a face of the polyhedron
- intersection of $P$ with hyperplane $c^\top x = OPT$

Fact
At a vertex, $n$ linearly independent constraints are satisfied with equality (a.k.a. tight)
An LP either has an optimal solution, or is unbounded or infeasible.
Fundamental Theorem of LP

If an LP in standard form has an optimal solution, then it has a vertex optimal solution.

Proof

Assume not, and take a non-vertex optimal solution $x$ with the maximum number of tight constraints. There is $y \neq 0$ such that $x \pm y$ are feasible.

$y$ is perpendicular to the objective function and the tight constraints at $x$. i.e. $c^\top y = 0$, and $a_i^\top y = 0$ whenever the $i$'th constraint is tight for $x$.

Can choose $y$ such that $y_j < 0$ for some $j$. Let $\alpha$ be the largest constant such that $x + \alpha y$ is feasible. Such an $\alpha$ exists. An additional constraint becomes tight at $x + \alpha y$, a contradiction.
Fundamental Theorem of LP

If an LP in standard form has an optimal solution, then it has a vertex optimal solution.

Proof

- Assume not, and take a non-vertex optimal solution $x$ with the maximum number of tight constraints.
Fundamental Theorem of LP
If an LP in standard form has an optimal solution, then it has a vertex optimal solution.

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- Assume not, and take a non-vertex optimal solution $x$ with the maximum number of tight constraints
- There is $y \neq 0$ s.t. $x \pm y$ are feasible
**Fundamental Theorem of LP**

If an LP in standard form has an optimal solution, then it has a vertex optimal solution.

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- Assume not, and take a non-vertex optimal solution $x$ with the maximum number of tight constraints.
- There is $y \neq 0$ s.t. $x \pm y$ are feasible.
- $y$ is perpendicular to the objective function and the tight constraints at $x$.
  - i.e. $c^T y = 0$, and $a_i^T y = 0$ whenever the $i$’th constraint is tight for $x$. 
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Proof

- Assume not, and take a non-vertex optimal solution \( x \) with the maximum number of tight constraints.
- There is \( y \neq 0 \) s.t. \( x \pm y \) are feasible.
- \( y \) is perpendicular to the objective function and the tight constraints at \( x \).
  - i.e. \( c^T y = 0 \), and \( a_i^T y = 0 \) whenever the \( i \)'th constraint is tight for \( x \).
- Can choose \( y \) s.t. \( y_j < 0 \) for some \( j \).
Fundamental Theorem of LP

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- Assume not, and take a non-vertex optimal solution \( x \) with the maximum number of tight constraints.
- There is \( y \neq 0 \) s.t. \( x \pm y \) are feasible.
- \( y \) is perpendicular to the objective function and the tight constraints at \( x \).
  - i.e. \( c^T y = 0 \), and \( a_i^T y = 0 \) whenever the \( i \)'th constraint is tight for \( x \).
- Can choose \( y \) s.t. \( y_j < 0 \) for some \( j \).
- Let \( \alpha \) be the largest constant such that \( x + \alpha y \) is feasible.
  - Such an \( \alpha \) exists.
Fundamental Theorem of LP

If an LP in standard form has an optimal solution, then it has a vertex optimal solution.

Proof

- Assume not, and take a non-vertex optimal solution $x$ with the maximum number of tight constraints.
- There is $y \neq 0$ s.t. $x \pm y$ are feasible.
- $y$ is perpendicular to the objective function and the tight constraints at $x$.
  - i.e. $c^T y = 0$, and $a_i^T y = 0$ whenever the $i$’th constraint is tight for $x$.
- Can choose $y$ s.t. $y_j < 0$ for some $j$.
- Let $\alpha$ be the largest constant such that $x + \alpha y$ is feasible.
  - Such an $\alpha$ exists.
- An additional constraint becomes tight at $x + \alpha y$, a contradiction.
Corollary

If an LP in standard form has an optimal solution, then there is an optimal solution with at most $m$ non-zero variables.

\[
\begin{align*}
\text{maximize} & \quad c^T x \\
\text{subject to} & \quad a_i^T x \leq b_i, \quad \text{for } i = 1, \ldots, m. \\
& \quad x_j \geq 0, \quad \text{for } j = 1, \ldots, n.
\end{align*}
\]

- e.g. for optimal production with $n$ products and $m$ raw materials, there is an optimal plan with at most $m$ products.
Linear Programming Duality

**Primal LP**

maximize \( c^T x \)

subject to \( Ax \leq b \)

**Dual LP**

minimize \( b^T y \)

subject to \( A^T y = c \)

\( y \geq 0 \)

- \( A \in \mathbb{R}^{m \times n}, c \in \mathbb{R}^n, b \in \mathbb{R}^m \)
- \( y_i \) is the dual variable corresponding to primal constraint \( A_i x \leq b_i \)
- \( A^T_j y \geq c_j \) is the dual constraint corresponding to primal variable \( x_j \)
Linear Programming Duality: Standard Form, and Visualization

Primal LP

maximize $c^T x$
subject to $Ax \leq b$
$x \geq 0$

Dual LP

minimize $y^T b$
subject to $A^T y \geq c$
$y \geq 0$
## Linear Programming Duality: Standard Form, and Visualization

### Primal LP

- maximize \( c^T x \)
- subject to \( A x \leq b \)
- \( x \geq 0 \)

### Dual LP

- minimize \( y^T b \)
- subject to \( A^T y \geq c \)
- \( y \geq 0 \)

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Linear Programming Duality: Standard Form, and Visualization

**Primal LP**

maximize \( c^T x \)
subject to \( Ax \leq b \)
\( x \geq 0 \)

**Dual LP**

minimize \( y^T b \)
subject to \( A^T y \geq c \)
\( y \geq 0 \)

\( y_i \) is the **dual variable** corresponding to primal constraint \( A_i x \leq b_i \)

\( A^T_j y \geq c_j \) is the **dual constraint** corresponding to primal variable \( x_j \)
Recall the Optimal Production problem from last lecture

- $n$ products, $m$ raw materials
- Every unit of product $j$ uses $a_{ij}$ units of raw material $i$
- There are $b_i$ units of material $i$ available
- Product $j$ yields profit $c_j$ per unit
- Facility wants to maximize profit subject to available raw materials

Primal LP

$$\text{max} \sum_{j=1}^{n} c_j x_j$$
$$\text{s.t.} \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i \in [m].$$
$$x_j \geq 0, \quad j \in [n].$$

Dual LP

$$\text{min} \sum_{i=1}^{m} b_i y_i$$
$$\text{s.t.} \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad j \in [n].$$
$$y_i \geq 0, \quad i \in [m].$$

Dual variable $y_i$ is a proposed price per unit of raw material $i$

Dual price vector is feasible if facility has incentive to sell materials

Buyer wants to spend as little as possible to buy materials
Primal LP

max \[ \sum_{j=1}^{n} c_j x_j \]

s.t. \[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \text{for } i \in [m]. \]

\[ x_j \geq 0, \quad \text{for } j \in [n]. \]
Interpretation 1: Economic Interpretation

**Primal LP**

\[
\begin{align*}
\text{max} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \text{for } i \in [m]. \\
& \quad x_j \geq 0, \quad \text{for } j \in [n].
\end{align*}
\]

**Dual LP**

\[
\begin{align*}
\text{min} & \quad \sum_{i=1}^{m} b_i y_i \\
\text{s.t.} & \quad \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad \text{for } j \in [n]. \\
& \quad y_i \geq 0, \quad \text{for } i \in [m].
\end{align*}
\]

Dual variable \( y_i \) is a proposed price per unit of raw material \( i \).

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Buyer wants to spend as little as possible to buy materials.
Interpretation 1: Economic Interpretation

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& \quad x_j \geq 0, \quad \text{for } j \in [n].
\end{align*}
\]

**Dual LP**

\[
\begin{align*}
\text{min} & \quad \sum_{i=1}^{m} b_i y_i \\
\text{s.t.} & \quad \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad \text{for } j \in [n]. \\
& \quad y_i \geq 0, \quad \text{for } i \in [m].
\end{align*}
\]

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Interpretation 1: Economic Interpretation

**Primal LP**

\[
\begin{align*}
\text{max } & \quad \sum_{j=1}^{n} c_j x_j \\
\text{s.t. } & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \text{for } i \in [m]. \\
& \quad x_j \geq 0, \quad \text{for } j \in [n].
\end{align*}
\]

**Dual LP**

\[
\begin{align*}
\text{min } & \quad \sum_{i=1}^{m} b_i y_i \\
\text{s.t. } & \quad \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad \text{for } j \in [n]. \\
& \quad y_i \geq 0, \quad \text{for } i \in [m].
\end{align*}
\]

- Dual variable \( y_i \) is a proposed **price** per unit of raw material \( i \)
- Dual price vector is feasible if facility has incentive to sell materials
- Buyer wants to spend as little as possible to buy materials
Consider the simple LP from last lecture

maximize \( x_1 + x_2 \)
subject to \( x_1 + 2x_2 \leq 2 \)
\( 2x_1 + x_2 \leq 2 \)
\( x_1, x_2 \geq 0 \)

We found that the optimal solution was at \( \left( \frac{2}{3}, \frac{2}{3} \right) \), with an optimal value of \( \frac{4}{3} \).
Interpretation 2: Finding the Best Upperbound

Consider the simple LP from last lecture

\[
\begin{align*}
\text{maximize} & \quad x_1 + x_2 \\
\text{subject to} & \quad x_1 + 2x_2 \leq 2 \\
& \quad 2x_1 + x_2 \leq 2 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

- We found that the optimal solution was at \( (\frac{2}{3}, \frac{2}{3}) \), with an optimal value of \( \frac{4}{3} \).
- What if, instead of finding the optimal solution, we sought to find an upperbound on its value by combining inequalities?
  - Each inequality implies an upper bound of 2
  - Multiplying each by \( \frac{1}{3} \) and summing gives \( x_1 + x_2 \leq \frac{4}{3} \).
Interpretation 2: Finding the Best Upperbound

- Multiplying each row $i$ by $y_i$ and summing gives the inequality

$$y^T Ax \leq y^T b$$
Interpretation 2: Finding the Best Upperbound

Multiplying each row $i$ by $y_i$ and summing gives the inequality

$$y^T Ax \leq y^T b$$

When $y^T A \geq c^T$, the right hand side of the inequality is an upper bound on $c^T x$ for every feasible $x$.

$$c^T x \leq y^T Ax \leq y^T b$$
Interpretation 2: Finding the Best Upperbound

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- Multiplying each row $i$ by $y_i$ and summing gives the inequality
  \[ y^T Ax \leq y^T b \]

- When $y^T A \geq c^T$, the right hand side of the inequality is an upper bound on $c^T x$ for every feasible $x$.
  \[ c^T x \leq y^T Ax \leq y^T b \]

- The dual LP can be thought of as trying to find the best upperbound on the primal that can be achieved this way.
Apply force field $c$ to a ball inside bounded polytope $Ax \leq b$. 
Interpretation 3: Physical Forces

- Apply force field $c$ to a ball inside bounded polytope $Ax \leq b$.
- Eventually, ball will come to rest against the walls of the polytope.
Apply force field $c$ to a ball inside bounded polytope $Ax \leq b$.

Eventually, ball will come to rest against the walls of the polytope.

Wall $a_i x \leq b_i$ applies some force $-y_i a_i$ to the ball
Interpretation 3: Physical Forces

- Apply force field $c$ to a ball inside bounded polytope $Ax \leq b$.
- Eventually, ball will come to rest against the walls of the polytope.
- Wall $a_i x \leq b_i$ applies some force $-y_i a_i$ to the ball.
- Since the ball is still, $c^T = \sum_i y_i a_i = y^T A$. 

Dual can be thought of as trying to minimize "work" $\sum_i y_i b_i$ to bring ball back to origin by moving polytope.
Interpretation 3: Physical Forces

- Apply force field \( c \) to a ball inside bounded polytope \( Ax \leq b \).
- Eventually, ball will come to rest against the walls of the polytope.
- Wall \( a_i x \leq b_i \) applies some force \(-y_i a_i\) to the ball
- Since the ball is still, \( c^T = \sum_i y_i a_i = y^T A \).
- Dual can be thought of as trying to minimize “work” \( \sum_i y_i b_i \) to bring ball back to origin by moving polytope
- We will see that, at optimality, only the walls adjacent to the ball push (Complementary Slackness)
Outline

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2. Duality and Its Interpretations
3. Properties of Duals
4. Weak and Strong Duality
5. Consequences of Duality
6. Uses and Examples of Duality
7. Solvability of LP
Duality is an Inversion

Primal LP

maximize $c^T x$
subject to $Ax \leq b$
$x \geq 0$

Dual LP

minimize $b^T y$
subject to $A^T y \geq c$
$y \geq 0$

Given a primal LP in standard form, the dual of its dual is itself.
Correspondance Between Variables and Constraints

**Primal LP**

\[
\text{max} \quad \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \text{for } i \in [m]. \\
x_j \geq 0, \quad \text{for } j \in [n].
\]

**Dual LP**

\[
\text{min} \quad \sum_{i=1}^{m} b_i y_i \\
\text{s.t.} \quad \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad \text{for } j \in [n]. \\
y_i \geq 0, \quad \text{for } i \in [m].
\]
Correspondance Between Variables and Constraints

**Primal LP**

\[
\begin{align*}
\text{max} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, & \text{for } i \in [m]. \\
& \quad x_j \geq 0, & \text{for } j \in [n]. \\
\end{align*}
\]

\(y_i:\)

**Dual LP**

\[
\begin{align*}
\text{min} & \quad \sum_{i=1}^{m} b_i y_i \\
\text{s.t.} & \quad \sum_{i=1}^{m} a_{ij} y_i \geq c_j, & \text{for } j \in [n]. \\
& \quad y_i \geq 0, & \text{for } i \in [m].
\end{align*}
\]

- The \(i\)’th primal constraint gives rise to the \(i\)’th dual variable \(y_i\)
Correspondance Between Variables and Constraints

**Primal LP**

\[
\text{max} \quad \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \text{for } i \in [m]. \\
x_j \geq 0, \quad \text{for } j \in [n].
\]

**Dual LP**

\[
\text{min} \quad \sum_{i=1}^{m} b_i y_i \\
\text{s.t.} \quad \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad \text{for } j \in [n]. \\
y_i \geq 0, \quad \text{for } i \in [m].
\]

- The \(i\)'th primal constraint gives rise to the \(i\)'th dual variable \(y_i\).
- The \(j\)'th primal variable \(x_j\) gives rise to the \(j\)'th dual constraint.
**Syntactic Rules**

**Primal LP**

\[
\begin{align*}
\text{max} \quad & c^\top x \\
\text{s.t.} \quad & y_i : \quad a_i x \leq b_i, \quad \text{for } i \in C_1. \\
& y_i : \quad a_i x = b_i, \quad \text{for } i \in C_2. \\
& x_j \geq 0, \quad \text{for } j \in D_1. \\
& x_j \in \mathbb{R}, \quad \text{for } j \in D_2.
\end{align*}
\]

**Dual LP**

\[
\begin{align*}
\text{min} \quad & b^\top y \\
\text{s.t.} \quad & x_j : \quad \bar{a}_j^\top y \geq c_j, \quad \text{for } j \in D_1. \\
& x_j : \quad \bar{a}_j^\top y = c_j, \quad \text{for } j \in D_2. \\
& y_i \geq 0, \quad \text{for } i \in C_1. \\
& y_i \in \mathbb{R}, \quad \text{for } i \in C_2.
\end{align*}
\]

**Rules of Thumb**

- Loose constraint (i.e. inequality) ⇒ tight dual variable (i.e. nonnegative)
- Tight constraint (i.e. equality) ⇒ loose dual variable (i.e. unconstrained)
## Weak Duality

**Primal LP**

- **maximize** \( c^T x \)
- **subject to** \( Ax \leq b \)
- \( x \geq 0 \)

**Dual LP**

- **minimize** \( b^T y \)
- **subject to** \( A^T y \geq c \)
- \( y \geq 0 \)

### Theorem (Weak Duality)

For every primal feasible \( x \) and dual feasible \( y \), we have \( c^T x \leq b^T y \).

### Corollary

- If primal and dual both feasible and bounded,
  \[ \text{OPT(Primal)} \leq \text{OPT(Dual)} \]
- If primal is unbounded, dual is infeasible
- If dual is unbounded, primal is infeasible
Weak Duality

**Primal LP**

maximize $c^T x$

subject to $Ax \leq b$

$x \geq 0$

**Dual LP**

minimize $b^T y$

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$y \geq 0$

---

**Theorem (Weak Duality)**

*For every primal feasible $x$ and dual feasible $y$, we have $c^T x \leq b^T y$.***

---

**Corollary**

*If $x$ is primal feasible, and $y$ is dual feasible, and $c^T x = b^T y$, then both are optimal.*
Economic Interpretation

If selling the raw materials is more profitable than making any individual product, then total money collected from sale of raw materials would exceed profit from production.
Interpretation of Weak Duality

Economic Interpretation
If selling the raw materials is more profitable than making any individual product, then total money collected from sale of raw materials would exceed profit from production.

Upperbound Interpretation
Self explanatory
Interpretation of Weak Duality

Economic Interpretation
If selling the raw materials is more profitable than making any individual product, then total money collected from sale of raw materials would exceed profit from production.

Upperbound Interpretation
Self explanatory

Physical Interpretation
Work required to bring ball back to origin by pulling polytope is at least potential energy difference between origin and primal optimum.
Proof of Weak Duality

Primal LP

maximize \( c^T x \)
subject to \( Ax \leq b \)
\( x \geq 0 \)

Dual LP

minimize \( b^T y \)
subject to \( A^T y \geq c \)
\( y \geq 0 \)

\[ c^T x \leq y^T Ax \leq y^T b \]
**Strong Duality**

**Primal LP**
- maximize $c^T x$
- subject to $Ax \leq b$
- $x \geq 0$

**Dual LP**
- minimize $b^T y$
- subject to $A^T y \geq c$
- $y \geq 0$

**Theorem (Strong Duality)**

*If either the primal or dual is feasible and bounded, then so is the other and $OPT(Primal) = OPT(Dual)$.***
Economic Interpretation

Buyer can offer prices for raw materials that would make facility indifferent between production and sale.
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Upperbound Interpretation
The method of scaling and summing inequalities yields a tight upperbound on the primal optimal value.
Interpretation of Strong Duality

**Economic Interpretation**
Buyer can offer prices for raw materials that would make facility indifferent between production and sale.

**Upperbound Interpretation**
The method of scaling and summing inequalities yields a tight upperbound on the primal optimal value.

**Physical Interpretation**
There is an assignment of forces to the walls of the polytope that brings ball back to the origin without wasting energy.
Informal Proof of Strong Duality

Recall the physical interpretation of duality

We found a primal and dual solution that are equal in value!
Informal Proof of Strong Duality

Recall the physical interpretation of duality
When ball is stationary at $x$, we expect force $c$ to be neutralized only by constraints that are tight. i.e. force multipliers $y \geq 0$ s.t.

- $y^t A = c$
- $y_i (b_i - a_i x) = 0$
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When ball is stationary at $x$, we expect force $c$ to be neutralized only by constraints that are tight. i.e. force multipliers $y \geq 0$ s.t.
- $y^T A = c$
- $y_i (b_i - a_i x) = 0$

$$y^T b - c^T x = y^T b - y^T A x = \sum_i y_i (b_i - a_i x) = 0$$

We found a primal and dual solution that are equal in value!
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3. Properties of Duals
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7. Solvability of LP
Complementary Slackness

Primal LP

maximize $c^T x$
subject to $Ax \leq b$
$x \geq 0$

Dual LP

minimize $y^T b$
subject to $A^T y \geq c$
$y \geq 0$

Let $s_i = (b - Ax)_i$ be the $i$'th primal slack variable
Let $t_j = (A^T y - c)_j$ be the $j$'th dual slack variable

Complementary Slackness

$x$ and $y$ are optimal if and only if $x_j t_j = 0$ for all $j = 1, ..., n$
$y_i s_i = 0$ for all $i = 1, ..., m$
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Complementary Slackness

**Primal LP**

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\begin{align*}
\text{maximize} & \quad c^\top x \\
\text{subject to} & \quad Ax \leq b \\
& \quad x \geq 0
\end{align*}
\]

**Dual LP**

\[
\begin{align*}
\text{minimize} & \quad y^\top b \\
\text{subject to} & \quad A^\top y \geq c \\
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- Let \( t_j = (A^\top y - c)_j \) be the \( j \)'th dual slack variable

**Complementary Slackness**

\( x \) and \( y \) are optimal if and only if
- \( x_j t_j = 0 \) for all \( j = 1, \ldots, n \)
- \( y_i s_i = 0 \) for all \( i = 1, \ldots, m \)
Economic Interpretation

Given an optimal primal production vector $x$ and optimal dual offer prices $y$,

- Facility produces only products for which it is indifferent between sale and production.
- Only raw materials that are binding constraints on production are priced greater than 0.
Physical Interpretation

Only walls adjacent to the balls equilibrium position push back on it.
Proof of Complementary Slackness

**Primal LP**

maximize \(c^\top x\)
subject to \(Ax \leq b\)
\(x \geq 0\)

**Dual LP**

minimize \(y^\top b\)
subject to \(A^\top y \geq c\)
\(y \geq 0\)

Can equivalently rewrite LP using slack variables
\[y^\top b - c^\top x = y^\top (Ax + s) - (y^\top A - t^\top) x = y^\top s + t^\top x\]

Gap between primal and dual objectives is 0 if and only if complementary slackness holds.
Proof of Complementary Slackness

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Gap between primal and dual objectives is 0 if and only if complementary slackness holds.
Recovering Primal from Dual

- Will encounter LPs where the dual is easier to solve than primal
- Complementary slackness allows us to recover the primal optimal from the dual optimal, and vice versa.

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Let \( y \) be dual optimal. By non-degeneracy:

- Exactly \( m \) of the \( m + n \) dual constraints are tight at \( y \)
- Exactly \( n \) dual constraints are loose

\( n \) loose dual constraints impose \( n \) tight primal constraints

Assuming non-degeneracy, solving the linear equation yields a unique primal optimum solution \( x \).
Recovering Primal from Dual

- Will encounter LPs where the dual is easier to solve than primal.
- Complementary slackness allows us to recover the primal optimal from the dual optimal, and vice versa.
  - Assuming non-degeneracy: At every vertex of primal [dual] there are exactly $n$ [m] tight constraints which are linearly independent.
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Primal LP

- $(n$ variables, $m + n$ constraints)
- maximize $c^T x$
- subject to $Ax \leq b$
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  - \( n \) loose dual constraints impose \( n \) tight primal constraints
  - Assuming non-degeneracy, solving the linear equation yields a unique primal optimum solution \( x \).

Consequences of Duality
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Uses of Duality in Algorithm Design

1. Gain structural insights
   - Dual of a problem gives a “different way of looking at it”.

2. As a benchmark; i.e. to certify (approximate) optimality
   - The primal/dual paradigm
   - A dual may be explicitly constructed by the algorithm, or as part of its analysis
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2. As a benchmark; i.e. to certify (approximate) optimality
   - The primal/dual paradigm
   - A dual may be explicitly constructed by the algorithm, or as part of its analysis

Let’s look at some duals and interpret them.
Given a directed network $G = (V, E)$ where edge $e$ has length $\ell_e \in \mathbb{R}_+$, find the minimum cost path from $s$ to $t$.
Shortest Path

Primal LP

\[
\begin{align*}
\text{min} & \quad \sum_{e \in E} \ell_e x_e \\
\text{s.t.} & \quad \sum_{e \to v} x_e - \sum_{v \to e} x_e = \delta_v, \quad \forall v \in V \\
& \quad x_e \geq 0, \quad \forall e \in E
\end{align*}
\]

Where \( \delta_v = -1 \) if \( v = s \), \( 1 \) if \( v = t \), and \( 0 \) otherwise.

Dual LP

\[
\begin{align*}
\text{max} & \quad y_t - y_s \\
\text{s.t.} & \quad y_v - y_u \leq \ell_e, \quad \forall (u, v) \in E
\end{align*}
\]
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\end{align*}
\]

Interpretation of Dual

Stretch \( s \) and \( t \) as far apart as possible, subject to edge lengths.

Uses and Examples of Duality
Given an undirected graph $G = (V, E)$, with weights $w_i$ for $i \in V$, find minimum-weight $S \subseteq V$ “covering” all edges.
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**Primal LP**

\[
\begin{align*}
\text{min} & \quad \sum_{i \in V} w_i x_i \\
\text{s.t.} & \quad x_i + x_j \geq 1, \quad \forall (i, j) \in E. \\
x & \geq 0
\end{align*}
\]

**Dual LP**

\[
\begin{align*}
\text{max} & \quad \sum_{e \in E} y_e \\
\text{s.t.} & \quad \sum_{e \in \Gamma(i)} y_e \leq w_i, \quad \forall i \in V. \\
y & \geq 0
\end{align*}
\]
Vertex Cover

Given an undirected graph $G = (V, E)$, with weights $w_i$ for $i \in V$, find minimum-weight $S \subseteq V$ “covering” all edges.

**Primal LP**

\[
\begin{align*}
\text{min} & \quad \sum_{i \in V} w_i x_i \\
\text{s.t.} & \quad x_i + x_j \geq 1, \quad \forall (i, j) \in E. \\
& \quad x \succeq 0
\end{align*}
\]

**Dual LP**

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\begin{align*}
\text{max} & \quad \sum_{e \in E} y_e \\
\text{s.t.} & \quad \sum_{e \in \Gamma(i)} y_e \leq w_i, \quad \forall i \in V. \\
& \quad y \succeq 0
\end{align*}
\]

**Interpretation of Dual**

Trying to “sell” coverage to edges at prices $y_e$.

- **Objective:** Maximize revenue
- **Feasible:** charge any neighborhood (of a vertex $i$) no more than it would cost them if they broke away and bought $i$ themselves
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maximize \quad c^T x \\
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\quad x \geq 0

- In the examples we have seen so far, the linear program is explicit
- I.e. the constraint matrix $A$, as well as rhs vector $b$ and objective $c$, are of polynomial size.
maximize $c^T x$
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- I.e. the constraint matrix $A$, as well as rhs vector $b$ and objective $c$, are of polynomial size.

**Theorem (Polynomial Solvability of Explicit LP)**

*There is a polynomial time algorithm for linear programming, when the linear program is represented explicitly.*

Originally using the ellipsoid algorithm, and more recently interior-point algorithms which are more efficient in practice.
Implicit Linear Programs

- These are linear programs in which the number of constraints is exponential (in the natural description of the input)
- These are useful as an analytical tool
- Can be solved in many cases!

\[
\begin{align*}
\min & \quad \sum_{e \in E} d_e x_e \\
\text{s.t.} & \quad x(\delta(S)) \geq 2, \quad \forall \emptyset \subset S \subset V. \\
& \quad x(\delta(v)) = 2, \quad \forall v \in V. \\
0 & \preceq x \preceq 1
\end{align*}
\]

Where \(\delta(S)\) denotes the edges going out of \(S \subseteq V\).
Implicit Linear Programs

- These are linear programs in which the number of constraints is exponential (in the natural description of the input)
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- E.g. Held-Karp relaxation for TSP

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\end{align*}
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maximize \( c^\top x \)
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Theorem (Polynomial Solvability of Implicit LP)
Consider a family \( \Pi \) of linear programming problems \( I = (A, b, c) \) admitting the following operations in polynomial time (in \( \langle I \rangle \) and \( n \)):
- A separation oracle for the polyhedron \( Ax \preceq b \)
- Explicit access to \( c \)

Moreover, assume that every \( \langle a_{ij} \rangle, \langle b_i \rangle, \langle c_j \rangle \) are at most \( \text{poly}(\langle I \rangle, n) \).
Then there is a polynomial time algorithm for \( \Pi \) (both primal and dual).

Separation oracle
An algorithm that takes as input \( x \in \mathbb{R}^n \), and either certifies \( Ax \preceq b \) or finds a violated constraint \( a_i x > b_i \).
E.g. of a Separation Oracle

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- Nontrivial part: given fixed \( x \) need to check whether \( x(\delta(S)) \geq 2 \) for all \( S \), else find such an \( S \) which violates this.
E.g. of a Separation Oracle

$$\min \sum_{e \in E} d_e x_e$$

$$\text{s.t.}$$

$$x(\delta(S)) \geq 2, \quad \forall \emptyset \subset S \subset V.$$  

$$x(\delta(v)) = 2, \quad \forall v \in V.$$  

$$0 \leq x \leq 1$$

- Nontrivial part: given fixed $x$, need to check whether $x(\delta(S)) \geq 2$ for all $S$, else find such an $S$ which violates this.
- Suffices to minimize $x(\delta(S))$ over all nonempty $S \subset V$. 

This is min-cut in a weighted graph, which we can solve.
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