Outline

1. Linear Programming
2. Application to Combinatorial Problems
3. Duality and Its Interpretations
4. Properties of Duals
Outline

1. Linear Programming
2. Application to Combinatorial Problems
3. Duality and Its Interpretations
4. Properties of Duals
The forefather of convex optimization problems, and the most ubiquitous.

- Best understood in that context
- But this is not a convex optimization class

Developed by Kantorovich during World War II (1939) for planning the Soviet army’s expenditures and returns. Kept secret.

Discovered a few years later by George Dantzig, who in 1947 developed the simplex method for solving linear programs.

John von Neumann developed LP duality in 1947, and applied it to game theory.

Polynomial-time solvable under fairly general conditions
- Ellipsoid method (Khachiyan 1979)
- Interior point methods (Karmarkar 1984).
minimize (or maximize) \( c^T x \)
subject to
\[ a_i^T x \leq b_i, \quad \text{for } i \in C^1. \]
\[ a_i^T x \geq b_i, \quad \text{for } i \in C^2. \]
\[ a_i^T x = b_i, \quad \text{for } i \in C^3. \]

- **Decision variables:** \( x \in \mathbb{R}^n \)
- **Parameters:**
  - \( c \in \mathbb{R}^n \) defines the linear objective function
  - \( a_i \in \mathbb{R}^n \) and \( b_i \in \mathbb{R} \) define the \( i \)'th constraint
Every LP can be transformed to either form

- minimizing $c^T x$ is equivalent to maximizing $-c^T x$
- inequality constraints can be flipped by multiplying by $-1$
- Each equality constraint can be replaced by two inequalities
- Unconstrained variable $x_j$ can be replaced by $x_j^+ - x_j^-$, where both $x_j^+$ and $x_j^-$ are constrained to be nonnegative.
Geometric View
A 2-D example

maximize \( x_1 + x_2 \)
subject to \( x_1 + 2x_2 \leq 2 \)
\( 2x_1 + x_2 \leq 2 \)
\( x_1, x_2 \geq 0 \)
**Economic Interpretation: Optimal Production**

- $n$ products, $m$ raw materials
- Every unit of product $j$ uses $a_{ij}$ units of raw material $i$
- There are $b_i$ units of material $i$ available
- Product $j$ yields profit $c_j$ per unit
- Facility wants to maximize profit subject to available raw materials

\[
\text{maximize} \quad c^T x \\
\text{subject to} \quad a_i^T x \leq b_i, \quad \text{for } i = 1, \ldots, m. \\
\quad x_j \geq 0, \quad \text{for } j = 1, \ldots, n.
\]
Hyperplane: The region defined by a linear equality
Halfspace: The region defined by a linear inequality $a_i^T x \leq b_i$.
Polyhedron: The intersection of a set of linear inequalities in Euclidean space
  Feasible region of an LP is a polyhedron
Polytope: A bounded polyhedron
  Equivalently: convex hull of a finite set of points
Vertex: A point $x$ is a vertex of polyhedron $P$ if $\exists y \neq 0$ with $x + y \in P$ and $x - y \in P$
Face of $P$: The intersection with $P$ of a hyperplane $H$ disjoint from the interior of $P
Fact

Feasible regions of LPs (i.e. polyhedrons) are convex
Basic Facts about LPs and Polytopes

**Fact**
Feasible regions of LPs (i.e. polyhedrons) are convex

**Fact**
Set of optimal solutions of an LP is convex
- In fact, a face of the polyhedron
- Intersection of $P$ with hyperplane $c^T x = OPT$
Basic Facts about LPs and Polytopes

Fact
Feasible regions of LPs (i.e. polyhedrons) are convex

Fact
Set of optimal solutions of an LP is convex
- In fact, a face of the polyhedron
- Intersection of $P$ with hyperplane $c^\top x = OPT$

Fact
At a vertex, $n$ linearly independent constraints are satisfied with equality (a.k.a. tight)
An LP either has an optimal solution, or is unbounded or infeasible.
Fundamental Theorem of LP

If an LP in standard form has an optimal solution, then it has a vertex optimal solution.

Proof

Assume not, and take a non-vertex optimal solution $x$ with the maximum number of tight constraints. There is $y \neq 0$ s.t. $x \pm y$ are feasible. $y$ is perpendicular to the objective function and the tight constraints at $x$. i.e. $c^\top y = 0$, and $a_i^\top y = 0$ whenever the $i$'th constraint is tight for $x$.

Can choose $y$ s.t. $y_j < 0$ for some $j$. Let $\alpha$ be the largest constant such that $x + \alpha y$ is feasible. Such an $\alpha$ exists. An additional constraint becomes tight at $x + \alpha y$, a contradiction.
Fundamental Theorem of LP

If an LP in standard form has an optimal solution, then it has a vertex optimal solution.

Proof

- Assume not, and take a non-vertex optimal solution $x$ with the maximum number of tight constraints
- There is $y \neq 0$ s.t. $x \pm y$ are feasible
- $y$ is perpendicular to the objective function and the tight constraints at $x$.
  - i.e. $c^T y = 0$, and $a_i^T y = 0$ whenever the $i$’th constraint is tight for $x$.
- Can choose $y$ s.t. $y_j < 0$ for some $j$
- Let $\alpha$ be the largest constant such that $x + \alpha y$ is feasible
  - Such an $\alpha$ exists
- An additional constraint becomes tight at $x + \alpha y$, a contradiction.
Corollary

If an LP in standard form has an optimal solution, then there is an optimal solution with at most $m$ non-zero variables.

maximize \( c^T x \)
subject to \( a_i^T x \leq b_i \), for \( i = 1, \ldots, m \).
\( x_j \geq 0 \), for \( j = 1, \ldots, n \).

e.g. for optimal production with $n$ products and $m$ raw materials, there is an optimal plan with at most $m$ products.
1. Linear Programming
2. Application to Combinatorial Problems
3. Duality and Its Interpretations
4. Properties of Duals
Linear programs often encode combinatorial problems either exactly or approximately.

Since our focus is on NP-hard problems, we encounter mostly the latter.

- An LP often relaxes the problem.
- Allows “better than optimal” solutions which are fractional.
Linear programs often encode combinatorial problems either exactly or approximately.

Since our focus is on NP-hard problems, we encounter mostly the latter.

- An LP often relaxes the problem.
- Allows “better than optimal” solutions which are fractional.

**Uses**

1. Rounding a solution of the LP
2. Analysis via primal/dual paradigm
Given a directed network \( G = (V, E) \) where edge \( e \) has length \( \ell_e \in \mathbb{R}_+ \), find the minimum cost path from \( s \) to \( t \).
Example: Shortest Path

Given a directed network $G = (V, E)$ where edge $e$ has length $\ell_e \in \mathbb{R}_+$, find the minimum cost path from $s$ to $t$.

![Diagram of a directed network with labeled edges and vertices](attachment:network_diagram.png)

**Shortest Path LP**

\[
\begin{align*}
\text{minimize} & \quad \sum_{e \in E} \ell_e x_e \\
\text{subject to} & \quad \sum_{e \to v} x_e - \sum_{v \to e} x_e = \delta_v, \quad \text{for } v \in V. \\
& \quad x_e \geq 0, \quad \text{for } e \in E.
\end{align*}
\]

Where $\delta_v = -1$ if $v = s$, $1$ if $v = t$, and $0$ otherwise.
Example: Vertex Cover

Given an undirected graph \( G = (V, E) \), with weights \( w_i \) for \( i \in V \), find minimum-weight \( S \subseteq V \) “covering” all edges.
Given an undirected graph $G = (V, E)$, with weights $w_i$ for $i \in V$, find minimum-weight $S \subseteq V$ “covering” all edges.

**Vertex Cover LP**

minimize $\sum_{i \in V} w_i x_i$

subject to $x_i + x_j \geq 1$, for $(i, j) \in E$.  
$x_i \geq 0$, for $i \in V$.  

Example: Knapsack

Given $n$ items with sizes $s_1, \ldots, s_n$ and values $v_1, \ldots, v_n$, and a knapsack of capacity $C$, find the maximum value set of items which fits in the knapsack.

$$\text{maximize } \sum_{i=1}^{n} v_i x_i$$

subject to

$$\sum_{i=1}^{n} s_i x_i \leq C$$

$x_i \leq 1$, for $i \in \{1, \ldots, n\}$.

$x_i \geq 0$, for $i \in \{1, \ldots, n\}$.
Example: Knapsack

Given \( n \) items with sizes \( s_1, \ldots, s_n \) and values \( v_1, \ldots, v_n \), and a knapsack of capacity \( C \), find the maximum value set of items which fits in the knapsack.

Knapsack LP

maximize \( \sum_{i=1}^{n} v_i x_i \)

subject to \( \sum_{i=1}^{n} s_i x_i \leq C \)

\( x_i \leq 1, \) for \( i \in [n] \).

\( x_i \geq 0, \) for \( i \in [n] \).
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### Linear Programming Duality

#### Primal LP

- **Maximize**: $c^T x$
- **Subject to**: $Ax \leq b$

#### Dual LP

- **Minimize**: $b^T y$
- **Subject to**: $A^T y = c$
  - $y \geq 0$

- $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$
- $y_i$ is the **dual variable** corresponding to primal constraint $A_i x \leq b_i$
- $A^T y \geq c_j$ is the **dual constraint** corresponding to primal variable $x_j$
Linear Programming Duality: Standard Form, and Visualization

**Primal LP**

maximize \( c^T x \)

subject to \( Ax \leq b \)

\( x \geq 0 \)

**Dual LP**

minimize \( y^T b \)

subject to \( A^T y \geq c \)

\( y \geq 0 \)
Linear Programming Duality: Standard Form, and Visualization

**Primal LP**

- maximize $c^T x$
- subject to $Ax \leq b$
- $x \geq 0$

**Dual LP**

- minimize $y^T b$
- subject to $A^T y \geq c$
- $y \geq 0$

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
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<tr>
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<td>$a_{31}$</td>
<td>$a_{32}$</td>
<td>$a_{33}$</td>
<td>$a_{34}$</td>
</tr>
<tr>
<td></td>
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Linear Programming Duality: Standard Form, and Visualization

**Primal LP**

\[
\begin{align*}
\text{maximize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b \\
& \quad x \geq 0
\end{align*}
\]

**Dual LP**

\[
\begin{align*}
\text{minimize} & \quad y^T b \\
\text{subject to} & \quad A^T y \geq c \\
& \quad y \geq 0
\end{align*}
\]

- \( y_i \) is the **dual variable** corresponding to primal constraint \( A_i x \leq b_i \)
- \( A^T_j y \geq c_j \) is the **dual constraint** corresponding to primal variable \( x_j \)
Interpretation 1: Economic Interpretation

Recall the Optimal Production problem

- $n$ products, $m$ raw materials
- Every unit of product $j$ uses $a_{ij}$ units of raw material $i$
- There are $b_i$ units of material $i$ available
- Product $j$ yields profit $c_j$ per unit
- Facility wants to maximize profit subject to available raw materials
Interpretation 1: Economic Interpretation

**Primal LP**

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\begin{align*}
\text{max} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \text{for } i \in [m]. \\
& \quad x_j \geq 0, \quad \text{for } j \in [n].
\end{align*}
\]

Dual variable \( y_i \) is a proposed price per unit of raw material \( i \).

Dual price vector is feasible if facility has incentive to sell materials.

Buyer wants to spend as little as possible to buy materials.
**Interpretation 1: Economic Interpretation**

**Primal LP**

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\begin{align*}
\text{max} & \quad \sum_{j=1}^{n} c_j x_j \\
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& \quad x_j \geq 0, \quad \text{for } j \in [n].
\end{align*}
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**Dual LP**

\[
\begin{align*}
\text{min} & \quad \sum_{i=1}^{m} b_i y_i \\
\text{s.t.} & \quad \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad \text{for } j \in [n]. \\
& \quad y_i \geq 0, \quad \text{for } i \in [m].
\end{align*}
\]

**Dual variable**

\[y_i\] is a proposed price per unit of raw material \(i\).

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Interpretation 1: Economic Interpretation

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Dual LP

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\begin{align*}
\text{min} & \quad \sum_{i=1}^{m} b_i y_i \\
\text{s.t.} & \quad \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad \text{for } j \in [n]. \\
& \quad y_i \geq 0, \quad \text{for } i \in [m].
\end{align*}
\]

Duality and Its Interpretations
Interpretation 1: Economic Interpretation

**Primal LP**

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\begin{align*}
\text{max} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \text{for } i \in [m]. \\
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**Dual LP**

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\text{min} & \quad \sum_{i=1}^{m} b_i y_i \\
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\]

- Dual variable \( y_i \) is a proposed price per unit of raw material \( i \)
- Dual price vector is feasible if facility has incentive to sell materials
- Buyer wants to spend as little as possible to buy materials
Recall the simple LP

\[
\begin{align*}
\text{maximize} & \quad x_1 + x_2 \\
\text{subject to} & \quad x_1 + 2x_2 \leq 2 \\
& \quad 2x_1 + x_2 \leq 2 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

We found that the optimal solution was at \( \left( \frac{2}{3}, \frac{2}{3} \right) \), with an optimal value of \( \frac{4}{3} \).
Recall the simple LP

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\begin{align*}
\text{maximize} & \quad x_1 + x_2 \\
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& \quad 2x_1 + x_2 \leq 2 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

- We found that the optimal solution was at \((\frac{2}{3}, \frac{2}{3})\), with an optimal value of \(\frac{4}{3}\).
- What if, instead of finding the optimal solution, we sought to find an upperbound on its value by combining inequalities?
  - Each inequality implies an upper bound of 2
  - Multiplying each by \(\frac{1}{3}\) and summing gives \(x_1 + x_2 \leq \frac{4}{3}\).
Interpretation 2: Finding the Best Upperbound

Multiplying each row $i$ by $y_i$ and summing gives the inequality

$$ y^T Ax \leq y^T b $$

When $y^T A \geq c^T$, the right hand side of the inequality is an upper bound on $c^T x$.

The dual LP can be thought of as trying to find the best upperbound on the primal that can be achieved this way.
Apply force field $c$ to a ball inside polytope $Ax \leq b$. 

Eventually, the ball will come to rest against the walls of the polytope. The wall $a_i x \leq b_i$ applies some force $-y_i a_i$ to the ball. Since the ball is still, $c^T = \sum_i y_i a_i = y^T A$.

Duality can be thought of as trying to minimize "work" $\sum_i y_i b_i$ to bring the ball back to origin by moving the polytope.

We will see that, at optimality, only the walls adjacent to the ball push (Complementary Slackness).
Interpretation 3: Physical Forces

- Apply force field $c$ to a ball inside polytope $Ax \leq b$.
- Eventually, ball will come to rest against the walls of the polytope.
Interpretation 3: Physical Forces

- Apply force field $c$ to a ball inside polytope $Ax \leq b$.
- Eventually, ball will come to rest against the walls of the polytope.
- Wall $a_i x \leq b_i$ applies some force $-y_i a_i$ to the ball.
Apply force field $c$ to a ball inside polytope $Ax \leq b$.
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Wall $a_i x \leq b_i$ applies some force $-y_i a_i$ to the ball.
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Interpretation 3: Physical Forces

- Apply force field \( c \) to a ball inside polytope \( Ax \leq b \).
- Eventually, ball will come to rest against the walls of the polytope.
- Wall \( a_i x \leq b_i \) applies some force \(-y_i a_i\) to the ball
- Since the ball is still, \( c^T = \sum_i y_i a_i = y^T A \).
- Dual can be thought of as trying to minimize “work” \( \sum_i y_i b_i \) to bring ball back to origin by moving polytope
- We will see that, at optimality, only the walls adjacent to the ball push (Complementary Slackness)
Outline

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Duality is an Inversion

Given a primal LP in standard form, the dual of its dual is itself.
Correspondance Between Variables and Constraints

**Primal LP**

max \[ \sum_{j=1}^{n} c_j x_j \]

s.t.

\[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \text{for } i \in [m]. \]

\[ x_j \geq 0, \quad \text{for } j \in [n]. \]

**Dual LP**

min \[ \sum_{i=1}^{m} b_i y_i \]

s.t.

\[ \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad \text{for } j \in [n]. \]

\[ y_i \geq 0, \quad \text{for } i \in [m]. \]
Correspondance Between Variables and Constraints

Primal LP

\[ \text{max} \quad \sum_{j=1}^{n} c_j x_j \]
\[ \text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad \text{for } i \in [m]. \]
\[ x_j \geq 0, \quad \text{for } j \in [n]. \]

Dual LP

\[ \text{min} \quad \sum_{i=1}^{m} b_i y_i \]
\[ \text{s.t.} \quad \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad \text{for } j \in [n]. \]
\[ y_i \geq 0, \quad \text{for } i \in [m]. \]

- The \( i \)'th primal constraint gives rise to the \( i \)'th dual variable \( y_i \)
### Correspondance Between Variables and Constraints

#### Primal LP

<table>
<thead>
<tr>
<th>Max</th>
<th>$\sum_{j=1}^{n} c_j x_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>s.t.</td>
<td>$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$, for $i \in [m]$.</td>
</tr>
<tr>
<td>$y_i$ :</td>
<td>$x_j \geq 0$, for $j \in [n]$.</td>
</tr>
</tbody>
</table>

#### Dual LP

<table>
<thead>
<tr>
<th>Min</th>
<th>$\sum_{i=1}^{m} b_i y_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>s.t.</td>
<td>$\sum_{i=1}^{m} a_{ij} y_i \geq c_j$, for $j \in [n]$.</td>
</tr>
<tr>
<td>$x_j$ :</td>
<td>$y_i \geq 0$, for $i \in [m]$.</td>
</tr>
</tbody>
</table>

- The $i$’th primal constraint gives rise to the $i$’th dual variable $y_i$.
- The $j$’th primal variable $x_j$ gives rise to the $j$’th dual constraint.
**Syntactic Rules**

### Primal LP

<table>
<thead>
<tr>
<th><strong>max</strong></th>
<th>$c^T x$</th>
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<tbody>
<tr>
<td><strong>s.t.</strong></td>
<td></td>
</tr>
<tr>
<td>$y_i :$</td>
<td>$a_i x \leq b_i$, for $i \in C_1$.</td>
</tr>
<tr>
<td>$y_i :$</td>
<td>$a_i x = b_i$, for $i \in C_2$.</td>
</tr>
<tr>
<td>$x_j \geq 0$, for $j \in D_1$.</td>
<td></td>
</tr>
<tr>
<td>$x_j \in \mathbb{R}$, for $j \in D_2$.</td>
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</tbody>
</table>

### Dual LP

<table>
<thead>
<tr>
<th><strong>min</strong></th>
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<tr>
<td><strong>s.t.</strong></td>
<td></td>
</tr>
<tr>
<td>$x_j :$</td>
<td>$\bar{a}_j^T y \geq c_j$, for $j \in D_1$.</td>
</tr>
<tr>
<td>$x_j :$</td>
<td>$\bar{a}_j^T y = c_j$, for $j \in D_2$.</td>
</tr>
<tr>
<td>$y_i \geq 0$, for $i \in C_1$.</td>
<td></td>
</tr>
<tr>
<td>$y_i \in \mathbb{R}$, for $i \in C_2$.</td>
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</tr>
</tbody>
</table>

### Rules of Thumb

- Loose constraint (i.e. inequality) $\Rightarrow$ tight dual variable (i.e. nonnegative)
- Tight constraint (i.e. equality) $\Rightarrow$ loose dual variable (i.e. unconstrained)