CS599: Convex and Combinatorial Optimization
Fall 2013
Lecture 3: Linear Programming Duality II

Instructor: Shaddin Dughmi
Announcements

- Today: wrap up linear programming
- Readings on website
1. Recall
2. Formal Proof of Strong Duality of LP
3. Consequences of Duality
4. More Examples of Duality
Weak and Strong Duality

**Primal LP**

maximize \( c^T x \)
subject to \( Ax \leq b \)
\( x \geq 0 \)

**Dual LP**

minimize \( b^T y \)
subject to \( A^T y \geq c \)
\( y \geq 0 \)

**Theorem (Weak Duality)**

\( \text{OPT}(\text{primal}) \leq \text{OPT}(\text{dual}) \).

**Theorem (Strong Duality)**

\( \text{OPT}(\text{primal}) = \text{OPT}(\text{dual}) \).
Informal Proof of Strong Duality

Recall the physical interpretation of duality

When the ball is stationary at $x$, we expect force $c$ to be neutralized only by constraints that are tight, i.e., force multipliers $y \geq 0$ such that $y^\top A = c$, $y^\top b - c^\top x = y^\top b - y^\top Ax = \sum_i y_i (b_i - a_i x) = 0$. We found a primal and dual solution that are equal in value!
Informal Proof of Strong Duality

Recall the physical interpretation of duality
When ball is stationary at $x$, we expect force $c$ to be neutralized only by constraints that are tight. i.e. force multipliers $y \geq 0$ s.t.

- $y^T A = c$
- $y_i (b_i - a_i x) = 0$
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When ball is stationary at $x$, we expect force $c$ to be neutralized only by constraints that are tight. i.e. force multipliers $y \geq 0$ s.t.

- $y^T A = c$
- $y_i (b_i - a_i x) = 0$

$$y^T b - c^T x = y^T b - y^T A x = \sum_i y_i (b_i - a_i x) = 0$$

We found a primal and dual solution that are equal in value!
Separating Hyperplane Theorem

If $A, B \subseteq \mathbb{R}^n$ are disjoint convex sets, then there is a hyperplane separating them. That is, there is $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $a^T x \leq b$ for every $x \in A$ and $a^T y \geq b$ for every $y \in B$. 
Definition

A convex cone is a convex subset of $\mathbb{R}^n$ which is closed under nonnegative scaling and convex combinations.

Definition

The convex cone generated by vectors $u_1, \ldots, u_m \in \mathbb{R}^n$ is the set of all nonnegative-weighted sums of these vectors (also known as conic combinations).

$$\text{Cone}(u_1, \ldots, u_m) = \left\{ \sum_{i=1}^{m} \alpha_i u_i : \alpha_i \geq 0 \forall i \right\}$$
The following follows from the separating hyperplane Theorem.

**Farkas’ Lemma**

Let $\mathcal{C}$ be the convex cone generated by vectors $u_1, \ldots, u_m \in \mathbb{R}^n$, and let $w \in \mathbb{R}^n$. Exactly one of the following is true:

- $w \in \mathcal{C}$
- There is $z \in \mathbb{R}^n$ such that $z \cdot u_i \leq 0$ for all $i$, and $z \cdot w \geq 0$.
Equivalently: Theorem of the Alternative

One of the following is true, where $U = [u_1, \ldots, u_m]$

- The system $Uy = w$, $y \geq 0$ has a solution
- The system $U^Tz \leq 0$, $z^Tw \geq 0$ has a solution.
Given $v$, by Farkas’ Lemma one of the following is true

1. The system $\begin{pmatrix} A^T \\ b^T \end{pmatrix} y = \begin{pmatrix} c \\ v \end{pmatrix}$, $y \geq 0$ has a solution.
   - $OPT(dual) \leq v$

2. The system $(A; b) z \leq 0$, $z^T \begin{pmatrix} c \\ v \end{pmatrix} \geq 0$ has a solution.
   - Let $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, where $z_1 \in \mathbb{R}^n$ and $z_2 \in \mathbb{R}$
   - Setting $x = -z_1/z_2$ gives $Ax \leq b$, $c^T x \geq v$.
   - $OPT(primal) \geq v$
1. Recall
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Complementary Slackness

**Primal LP**
- maximize \( c^T x \)
- subject to \( Ax \leq b \)
- \( x \geq 0 \)

**Dual LP**
- minimize \( y^T b \)
- subject to \( A^T y \geq c \)
- \( y \geq 0 \)

Let \( s_i = (b - Ax)_i \) be the \( i \)'th primal slack variable
Let \( t_j = (A^T y - c)_j \) be the \( j \)'th dual slack variable

Complementary Slackness

\( x \) and \( y \) are optimal if and only if
\[
\begin{align*}
  x_j t_j &= 0 \\
  y_i s_i &= 0
\end{align*}
\] for all \( j = 1, \ldots, n \) and \( i = 1, \ldots, m \)
Complementary Slackness

Primal LP

maximize \( c^T x \)
subject to \( Ax \leq b \)
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Dual LP

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Complementary Slackness

**Primal LP**

maximize \( c^\top x \)  
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### Complementary Slackness

\( x \) and \( y \) are optimal if and only if
- \( x_j t_j = 0 \) for all \( j = 1, \ldots, n \)
- \( y_i s_i = 0 \) for all \( i = 1, \ldots, m \)
Interpretation of Complementary Slackness

**Economic Interpretation**

Given an optimal primal production vector \( x \) and optimal dual offer prices \( y \),

- Facility produces only products for which it is indifferent between sale and production.
- Only raw materials that are binding constraints on production are priced greater than \( 0 \).
Interpretation of Complementary Slackness

Physical Interpretation

Only walls adjacent to the balls equilibrium position push back on it.
Proof of Complementary Slackness

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Consequences of Duality
Proof of Complementary Slackness

**Primal LP**

- Maximize: \( c^\top x \)
- Subject to:
  - \( Ax + s = b \)
  - \( x \geq 0 \)
  - \( s \geq 0 \)

**Dual LP**

- Minimize: \( y^\top b \)
- Subject to:
  - \( A^\top y - t = c \)
  - \( y \geq 0 \)
  - \( t \geq 0 \)

- Can equivalently rewrite LP using slack variables

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Consequences of Duality
Proof of Complementary Slackness

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maximize \( c^\top x \)
subject to
\[ Ax + s = b \]
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Dual LP

minimize \( y^\top b \)
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\[ t \geq 0 \]

Can equivalently rewrite LP using slack variables

\[
y^\top b - c^\top x = y^\top (Ax + s) - (y^\top A - t^\top)x = y^\top s + t^\top x
\]
Proof of Complementary Slackness

Primal LP

maximize $c^T x$
subject to $Ax + s = b$
$x \geq 0$
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subject to $A^T y - t = c$
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Can equivalently rewrite LP using slack variables

$$y^T b - c^T x = y^T (Ax + s) - (y^T A - t^T) x = y^T s + t^T x$$

Gap between primal and dual objectives is 0 if and only if complementary slackness holds.
Recovering Primal from Dual

- Will encounter LPs where the dual is easier to solve than primal
- Complementary slackness allows us to recover the primal optimal from the dual optimal, and vice versa.

**Primal LP**
- \( n \) variables, \( m+n \) constraints
- Maximize \( c^\top x \)
- Subject to \( Ax \leq b \), \( x \geq 0 \)

**Dual LP**
- \( m \) variables, \( m+n \) constraints
- Minimize \( y^\top b \)
- Subject to \( A^\top y \geq c \), \( y \geq 0 \)

Let \( y \) be dual optimal. By non-degeneracy:
- Exactly \( m \) of the \( m+n \) dual constraints are tight at \( y \)
- Exactly \( n \) dual constraints are loose
- \( n \) loose dual constraints impose \( n \) tight primal constraints

Assuming non-degeneracy, solving the linear equation yields a unique primal optimum solution \( x \).
Recovering Primal from Dual

- Will encounter LPs where the dual is easier to solve than primal
- Complementary slackness allows us to recover the primal optimal from the dual optimal, and vice versa.
  - Assuming non-degeneracy: Every vertex of primal [dual] is the solution of exactly $n$ [$m$] tight constraints.
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\( n \) variables, \( m + n \) constraints

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- Exactly \( m \) of the \( m + n \) dual constraints are tight at \( y \)
- Exactly \( n \) dual constraints are loose
- \( n \) loose dual constraints impose \( n \) tight primal constraints
- Assuming non-degeneracy, solving the linear equation yields a unique primal optimum solution \( x \).
### Sensitivity Analysis

#### Primal LP

- **Maximize**: \( c^T x \)
- **Subject to**:
  - \( Ax \leq b \)
  - \( x \geq 0 \)

#### Dual LP

- **Minimize**: \( y^T b \)
- **Subject to**:
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  - \( y \geq 0 \)

Sometimes, we want to examine how the optimal value of our LP changes with its parameters \( c \) and \( b \).
Sensitivity Analysis

Sometimes, we want to examine how the optimal value of our LP changes with its parameters $c$ and $b$.

Let $OPT = OPT(A, c, b)$ be the optimal value of the above LP. Let $x$ and $y$ be the primal and dual optima.

- $\frac{\partial OPT}{\partial c_j} = x_j$ when $x$ is the unique primal optimum.
- $\frac{\partial OPT}{\partial b_i} = y_i$ when $y$ is the unique dual optimum.
Sensitivity Analysis

**Primal LP**

maximize $c^T x$

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Sometimes, we want to examine how the optimal value of our LP changes with its parameters $c$ and $b$

**Economic Interpretation of Sensitivity Analysis**

- A small increase $\delta$ in $c_j$ increases profit by $\delta \cdot x_j$
- A small increase $\delta$ in $b_i$ increases profit by $\delta \cdot y_i$
- $y_i$ measures the “marginal value” of resource $i$ for production
Outline

1. Recall
2. Formal Proof of Strong Duality of LP
3. Consequences of Duality
4. More Examples of Duality
Shortest Path

Given a directed network $G = (V, E)$ where edge $e$ has length $\ell_e \in \mathbb{R}_+$, find the minimum cost path from $s$ to $t$. 

![Graph](image)

Where $\delta_v = -1$ if $v = s$, 1 if $v = t$, and 0 otherwise.

Interpretation of Dual

Stretch $s$ and $t$ as far apart as possible, subject to edge lengths.
Shortest Path

Primal LP
\[
\begin{align*}
\text{min} & \sum_{e \in E} \ell_e x_e \\
\text{s.t.} & \sum_{e \rightarrow v} x_e - \sum_{v \rightarrow e} x_e = \delta_v, \quad \forall v \in V. \\
& x_e \geq 0, \quad \forall e \in E.
\end{align*}
\]

Where \( \delta_v = -1 \) if \( v = s \), \( 1 \) if \( v = t \), and \( 0 \) otherwise.

Dual LP
\[
\begin{align*}
\text{max} & \quad y_t - y_s \\
\text{s.t.} & \quad y_v - y_u \leq \ell_e, \quad \forall (u, v) \in E.
\end{align*}
\]
Primal LP
\[
\min \sum_{e \in E} \ell_e x_e \\
\text{s.t.} \\
\sum_{e \rightarrow v} x_e - \sum_{v \rightarrow e} x_e = \delta_v, \quad \forall v \in V. \\
x_e \geq 0, \quad \forall e \in E.
\]

Where \( \delta_v = -1 \) if \( v = s \), \( 1 \) if \( v = t \), and \( 0 \) otherwise.

Dual LP
\[
\max y_t - y_s \\
\text{s.t.} \\
y_v - y_u \leq \ell_e, \quad \forall (u, v) \in E.
\]

Interpretation of Dual
Stretch \( s \) and \( t \) as far apart as possible, subject to edge lengths.
Maximum Weighted Bipartite Matching

Set $B$ of buyers, and set $G$ of goods. Buyer $i$ has value $w_{ij}$ for good $j$, and interested in at most one good. Find maximum value assignment of goods to buyers.
Maximum Weighted Bipartite Matching

**Primal LP**

\[
\text{max } \sum_{i,j} w_{ij} x_{ij}
\]

s.t.

\[
\sum_{j \in G} x_{ij} \leq 1, \quad \forall i \in B.
\]

\[
\sum_{i \in B} x_{ij} \leq 1, \quad \forall j \in G.
\]

\[
x_{ij} \geq 0, \quad \forall i \in B, j \in G.
\]

**Dual LP**

\[
\text{min } \sum_{i \in B} u_i + \sum_{j \in G} p_j
\]

s.t.

\[
u_i + p_j \geq w_{ij}, \quad \forall i \in B, j \in G.
\]

\[
u_i \geq 0, \quad \forall i \in B.
\]

\[
p_j \geq 0, \quad \forall j \in G.
\]

Interpretation of Dual:

- \( p_j \) is price of good \( j \)
- \( u_i \) is utility of buyer \( i \)

Complementary Slackness: each buyer grabs his favorite good given prices.
Maximum Weighted Bipartite Matching

**Primal LP**

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\begin{align*}
\text{max} & \quad \sum_{i,j} w_{ij} x_{ij} \\
\text{s.t.} & \quad \sum_{j \in G} x_{ij} \leq 1, \quad \forall i \in B. \\
& \quad \sum_{i \in B} x_{ij} \leq 1, \quad \forall j \in G. \\
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**Dual LP**

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\begin{align*}
\text{min} & \quad \sum_{i \in B} u_i + \sum_{j \in G} p_j \\
\text{s.t.} & \quad u_i + p_j \geq w_{ij}, \quad \forall i \in B, j \in G. \\
& \quad u_i \geq 0, \quad \forall i \in B. \\
& \quad p_j \geq 0, \quad \forall j \in G.
\end{align*}
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**Interpretation of Dual**

- \( p_j \) is price of good \( j \)
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More Examples of Duality
Rock-Paper-Scissors

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- Two players, row and column
- Game described by matrix $A$
- When row player plays pure strategy $i$ and column player plays pure strategy $j$, row player pays column player $A_{ij}$
2-Player Zero-Sum Games

Rock-Paper-Scissors

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- Two players, row and column
- Game described by matrix $A$
- When row player plays pure strategy $i$ and column player plays pure strategy $j$, row player pays column player $A_{ij}$
- **Mixed Strategy**: distribution over pure strategies
Two players, row and column

Game described by matrix $A$

When row player plays pure strategy $i$ and column player plays pure strategy $j$, row player pays column player $A_{ij}$

**Mixed Strategy**: distribution over pure strategies

Assume players know each other’s mixed strategies but not coin flips
2-Player Zero-Sum Games

Assume row player moves first with distribution $y \in \Delta_m$
- Loss as a function of column’s strategy given by $y^T A$
- A best response by column is pure strategy $j$ maximizing $(y^T A)_j$

More Examples of Duality
2-Player Zero-Sum Games

- Assume row player moves first with distribution $y \in \Delta_m$
- Loss as a function of column’s strategy given by $y^\top A$
- A best response by column is pure strategy $j$ maximizing $(y^\top A)_j$

Row Moves First

$$\begin{align*}
\min \ & \max_j (y^\top A)_j \\
\text{s.t.} \ & \\
& \sum_{i=1}^m y_i = 1 \\
& \ y \geq \vec{0}
\end{align*}$$
2-Player Zero-Sum Games

- Assume row player moves first with distribution $y \in \Delta_m$
  - Loss as a function of column’s strategy given by $y^T A$
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Row Moves First

$$\min u$$
$$\text{s.t.}$$
$$u\mathbf{1} - y^T A \geq \mathbf{0}$$
$$\sum_{i=1}^m y_i = 1$$
$$y \geq \mathbf{0}$$
2-Player Zero-Sum Games

- Assume row player moves first with distribution $y \in \Delta_m$
  - Loss as a function of column’s strategy given by $y^T A$
  - A best response by column is pure strategy $j$ maximizing $(y^T A)_j$
  - Similarly when column moves first

Row Moves First

$$\begin{align*}
\text{min } u \\
\text{s.t. } u \mathbf{1} - y^T A &\geq \mathbf{0} \\
\sum_{i=1}^m y_i &= 1 \\
y &\geq \mathbf{0}
\end{align*}$$

Column Moves First

$$\begin{align*}
\text{max } v \\
\text{s.t. } v \mathbf{1} - Ax &\leq \mathbf{0} \\
\sum_{j=1}^n x_j &= 1 \\
x &\geq \mathbf{0}
\end{align*}$$
2-Player Zero-Sum Games

- Assume row player moves first with distribution \( y \in \Delta_m \)
  - Loss as a function of column’s strategy given by \( y^\top A \)
  - A best response by column is pure strategy \( j \) maximizing \( (y^\top A)_j \)
  - Similarly when column moves first

Row Moves First

\[
\begin{align*}
\min & \quad u \\
\text{s.t.} & \quad u\mathbf{1} - y^\top A \geq \mathbf{0} \\
& \quad \sum_{i=1}^m y_i = 1 \\
& \quad y \geq 0
\end{align*}
\]

Column Moves First

\[
\begin{align*}
\max & \quad v \\
\text{s.t.} & \quad v\mathbf{1} - Ax \leq \mathbf{0} \\
& \quad \sum_{j=1}^n x_j = 1 \\
& \quad x \geq 0
\end{align*}
\]

These two optimization problems are LP Duals!
Weak Duality

- $u^* \geq v^*$
- Zero sum games have a second mover advantage
Duality and Zero Sum Games

**Weak Duality**
- \( u^* \geq v^* \)
- Zero sum games have a second mover advantage

**Strong Duality (Minimax Theorem)**
- \( u^* = v^* \)
- There is no second or first mover advantage in zero sum games with mixed strategies
- Each player can guarantee \( u^* = v^* \) regardless of other's strategy.
- \( y^*, x^* \) are simultaneously best responses to each other (Nash Equilibrium)
Duality and Zero Sum Games

Weak Duality

- $u^* \geq v^*$
- Zero sum games have a second mover advantage

Strong Duality (Minimax Theorem)

- $u^* = v^*$
- There is no second or first mover advantage in zero sum games with mixed strategies
- Each player can guarantee $u^* = v^*$ regardless of other’s strategy.
- $y^*, x^*$ are simultaneously best responses to each other (Nash Equilibrium)

Complementary Slackness

$x^*$ randomizes over pure best responses to $y^*$, and vice versa.
Saddle Point Interpretation

Consider the matching pennies game

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- Unique equilibrium: each player randomizes uniformly
- If row player deviates, he pays out more
- If column player deviates, he gets paid less
Unique equilibrium: each player randomizes uniformly
If row player deviates, he pays out more
If column player deviates, he gets paid less
Begin Convex Optimization Background: Convex Sets