Announcements

- Yu’s office hours changed to Friday 4pm-5pm
- Solutions to HW1 should be posted soon.
- HW2 coming soon
- This week: Convex Optimization Duality
  - Read all of B&V Chapter 5.
1. The Lagrange Dual Problem

2. Duality
Recall: Optimization Problem in Standard Form

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad \text{for } i = 1, \ldots, m. \\
                & \quad h_i(x) = 0, \quad \text{for } i = 1, \ldots, k.
\end{align*}
\]

- For convex optimization problems in standard form, \( f_i \) is convex and \( h_i \) is affine.
- Let \( D \) denote the domain of all these functions (i.e. when their value is finite)
Recall: Optimization Problem in Standard Form

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad \text{for } i = 1, \ldots, m. \\
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\end{align*}
\]

- For convex optimization problems in standard form, \( f_i \) is convex and \( h_i \) is affine.
- Let \( \mathcal{D} \) denote the domain of all these functions (i.e. when their value is finite)

This Lecture + Next
We will develop duality theory for convex optimization problems, generalizing linear programming duality.
We have already seen the standard form LP below:

**Maximize** $c^T x$

Subject to:

$Ax \leq b$

$x \geq 0$

**Minimize** $-c^T x$

Subject to:

$Ax - b \leq 0$

$-x \leq 0$
Running Example: Linear Programming

We have already seen the standard form LP below

\[
\begin{align*}
\text{maximize} & \quad c^\top x \\
\text{subject to} & \quad Ax \leq b \\
& \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{minimize} & \quad -c^\top x \\
\text{subject to} & \quad Ax - b \leq 0 \\
& \quad -x \leq 0
\end{align*}
\]

Along the way, we will recover the following standard form dual

\[
\begin{align*}
\text{minimize} & \quad y^\top b \\
\text{subject to} & \quad A^\top y \geq c \\
& \quad y \geq 0
\end{align*}
\]
minimize \( f_0(x) \)
subject to \( f_i(x) \leq 0, \) for \( i = 1, \ldots, m. \)
\( h_i(x) = 0, \) for \( i = 1, \ldots, k. \)

Basic idea of Lagrangian duality is to relax/soften the constraints by replacing each with a linear “penalty term” or “cost” in the objective.
The Lagrangian

minimize \( f_0(x) \)
subject to \( f_i(x) \leq 0, \) for \( i = 1, \ldots, m. \)
\( h_i(x) = 0, \) for \( i = 1, \ldots, k. \)

Basic idea of Lagrangian duality is to relax/soften the constraints by replacing each with a linear “penalty term” or “cost” in the objective.

The Lagrangian Function

\[
L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{k} \nu_i h_i(x)
\]

- \( \lambda_i \) is Lagrange Multiplier for \( i \)'th inequality constraint
  - Required to be nonnegative
- \( \nu_i \) is Lagrange Multiplier for \( i \)'th equality constraint
  - Allowed to be of arbitrary sign
The Lagrange Dual Function

minimize \( f_0(x) \)
subject to \( f_i(x) \leq 0, \text{ for } i = 1, \ldots, m. \)
\( h_i(x) = 0, \text{ for } i = 1, \ldots, k. \)

The Lagrange dual function gives the optimal value of the primal problem subject to the softened constraints.
The Lagrange Dual Function

minimize \( f_0(x) \)
subject to \( f_i(x) \leq 0, \) for \( i = 1, \ldots, m. \)
\( h_i(x) = 0, \) for \( i = 1, \ldots, k. \)

The Lagrange dual function gives the optimal value of the primal problem subject to the softened constraints.

\[
\begin{align*}
g(\lambda, \nu) &= \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) \\
&= \inf_{x \in \mathcal{D}} \left( f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{k} \nu_i h_i(x) \right)
\end{align*}
\]

- Observe: \( g \) is a concave function of the Lagrange multipliers.
- We will see: It's quite common for the Lagrange dual to be unbounded \((-\infty)\) for some \( \lambda \) and \( \nu \).
- By convention, domain of \( g \) is \((\lambda, \nu)\) s.t. \( g(\lambda, \nu) > -\infty \).
Langrange Dual of LP

minimize \(-c^\top x\)
subject to
\[Ax - b \leq 0\]
\[-x \leq 0\]

First, the Lagrangian function

\[L(x, \lambda) = -c^\top x + \lambda_1^\top (Ax - b) - \lambda_2^\top x\]
\[= (A^\top \lambda_1 - c - \lambda_2)^\top x - \lambda_1^\top b\]
Langrange Dual of LP

minimize $-c^T x$
subject to $Ax - b \leq 0$
$-x \leq 0$

First, the Lagrangian function

$$L(x, \lambda) = -c^T x + \lambda_1^T (Ax - b) - \lambda_2^T x$$
$$= (A^T \lambda_1 - c - \lambda_2)^T x - \lambda_1^T b$$

And the Lagrange Dual

$$g(\lambda) = \inf_x L(x, \lambda)$$
$$= \begin{cases} -\infty & \text{if } A^T \lambda_1 - c - \lambda_2 \neq 0 \\ -\lambda_1^T b & A^T \lambda_1 - c - \lambda_2 = 0 \end{cases}$$
minimize $-c^\top x$
subject to $Ax - b \leq 0$
$-x \leq 0$

First, the Lagrangian function

$$L(x, \lambda) = -c^\top x + \lambda_1^\top (Ax - b) - \lambda_2^\top x$$
$$= (A^\top \lambda_1 - c - \lambda_2)^\top x - \lambda_1^\top b$$

And the Lagrange Dual

$$g(\lambda) = \inf_x L(x, \lambda)$$
$$= \begin{cases} -\infty & \text{if } A^\top \lambda_1 - c - \lambda_2 \neq 0 \\ -\lambda_1^\top b & A^\top \lambda_1 - c - \lambda_2 = 0 \end{cases}$$

So we restrict the domain of $g$ to $\lambda$ satisfying $A^\top \lambda_1 - c - \lambda_2 = 0$
Interpretation: “Soft” Lower Bound

\[
\begin{align*}
\text{min} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad \text{for } i = 1, \ldots, m. \\
& \quad h_i(x) = 0, \quad \text{for } i = 1, \ldots, k.
\end{align*}
\]

The Lagrange Dual Function

\[
g(\lambda, \nu) = \inf_{x \in D} L(x, \lambda, \nu) = \inf_{x \in D} \left( f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{k} \nu_i h_i(x) \right)
\]
Interpretation: “Soft” Lower Bound

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\begin{align*}
\min & \quad f_0(x) \\
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The Lagrange Dual Function

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g(\lambda, \nu) = \inf_{x \in D} L(x, \lambda, \nu) = \inf_{x \in D} \left( f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{k} \nu_i h_i(x) \right)
\]

Fact

\(g(\lambda, \nu)\) is a lowerbound on OPT(primal) for every \(\lambda \succeq 0\) and \(\nu \in \mathbb{R}^k\).
Interpretation: “Soft” Lower Bound

\[
\begin{align*}
\text{min} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad \text{for } i = 1, \ldots, m. \\
& \quad h_i(x) = 0, \quad \text{for } i = 1, \ldots, k.
\end{align*}
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The Lagrange Dual Function

\[
g(\lambda, \nu) = \inf_{x \in D} L(x, \lambda, \nu) = \inf_{x \in D} \left( f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{k} \nu_i h_i(x) \right)
\]

Fact

\(g(\lambda, \nu)\) is a lowerbound on \(OPT(\text{primal})\) for every \(\lambda \succeq 0\) and \(\nu \in \mathbb{R}^k\).

Proof

- Every primal feasible \(x\) incurs nonpositive penalty by \(L(x, \lambda, \nu)\)
- Therefore, \(L(x^*, \lambda, \nu) \leq f_0(x^*)\)
- So \(g(\lambda, \nu) \leq f_0(x^*) = OPT(Primal)\)
Interpretation: “Soft” Lower Bound

\[
\begin{align*}
\min & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad \text{for } i = 1, \ldots, m. \\
& \quad h_i(x) = 0, \quad \text{for } i = 1, \ldots, k.
\end{align*}
\]

The Lagrange Dual Function

\[
g(\lambda, \nu) = \inf_{x \in D} L(x, \lambda, \nu) = \inf_{x \in D} \left( f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{k} \nu_i h_i(x) \right)
\]

Interpretation

- A “hard” feasibility constraint can be thought of as imposing a penalty of \(+\infty\) if violated.
- Lagrangian imposes a “soft” linear penalty for violating a constraint, and a reward for slack.
- Lagrange dual finds the optimal subject to these soft constraints.
Interpretation: “Soft” Lower Bound
Interpretation: Geometric

Most easily visualized in the presence of a single inequality constraint

\[
\begin{align*}
&\text{minimize} & & f_0(x) \\
&\text{subject to} & & f_1(x) \leq 0
\end{align*}
\]

Let \( G \) be attainable constraint/objective function value tuples

- i.e. \((u, t) \in G\) if there is an \( x \) such that \( f_1(x) = u \) and \( f_0(x) = t \)
- \( p^* = \inf \{ t : (u, t) \in G, u \leq 0 \} \)
- \( g(\lambda) = \inf \{ \lambda u + t : (u, t) \in G \} \)
Interpretation: Geometric

Most easily visualized in the presence of a single inequality constraint

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_1(x) \leq 0
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- Let \( G \) be attainable constraint/objective function value tuples
  - i.e. \((u, t) \in G\) if there is an \( x \) such that \( f_1(x) = u \) and \( f_0(x) = t \)
- \( p^* = \inf \{ t : (u, t) \in G, u \leq 0 \} \)
- \( g(\lambda) = \inf \{ \lambda u + t : (u, t) \in G \} \)
- \( \lambda u + t = g(\lambda) \) is a supporting hyperplane to \( G \) pointing northeast
- Must intersect vertical axis below \( p^* \)
- Therefore \( g(\lambda) \leq p^* \)
The Lagrange Dual Problem

This is the problem of finding the best lower bound on $\text{OPT}(\text{primal})$ implied by the Lagrange dual function

$$\text{maximize} \quad g(\lambda, \nu)$$
$$\text{subject to} \quad \lambda \succeq 0$$

Note: this is a convex optimization problem, regardless of whether primal problem was convex

By convention, sometimes we add “dual feasibility” constraints to impose “nontrivial” lowerbounds (i.e. $g(\lambda, \nu) \geq -\infty$)

$$(\lambda^*, \nu^*)$$ solving the above are referred to as the dual optimal solution
maximize \( c^T x \) 
subject to \( Ax \leq b \) 
\( x \geq 0 \)

minimize \(-c^T x\)
subject to \( Ax - b \leq 0 \)
\(-x \leq 0\)

Recall

Our Lagrange dual function for the above LP (to the right), defined over the domain \( A^T \lambda_1 - c - \lambda_2 = 0 \).

\[
g(\lambda) = -\lambda_1^T b
\]
Langrange Dual Problem of LP

maximize \( c^\top x \)
subject to \( Ax \leq b \)
\( x \geq 0 \)

minimize \(-c^\top x\)
subject to \( Ax - b \leq 0 \)
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Recall

Our Lagrange dual function for the above LP (to the right), defined over the domain \( A^\top \lambda_1 - c - \lambda_2 = 0 \).

\[
g(\lambda) = -\lambda_1^\top b
\]

The Lagrange dual problem can then be written as

maximize \(-\lambda_1^\top b\)
subject to \( A^\top \lambda_1 - c - \lambda_2 = 0 \)
\( \lambda \geq 0 \)
Langrange Dual Problem of LP

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\begin{align*}
\text{maximize} & \quad c^\top x \\
\text{subject to} & \quad Ax \leq b \\
& \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{minimize} & \quad -c^\top x \\
\text{subject to} & \quad Ax - b \leq 0 \\
& \quad -x \leq 0
\end{align*}
\]

Recall

Our Lagrange dual function for the above LP (to the right), defined over the domain \( A^\top \lambda_1 - c - \lambda_2 = 0 \).

\[g(\lambda) = -\lambda_1^\top b\]

The Lagrange dual problem can then be written as

\[
\begin{align*}
\text{maximize} & \quad -\lambda_1^\top b \\
\text{subject to} & \quad A^\top \lambda_1 - c - \lambda_2 = 0 \\
& \quad A^\top \lambda_1 \geq c \\
& \quad \lambda \geq 0
\end{align*}
\]
### Langrange Dual Problem of LP

**maximize** \( c^T x \) 
subject to \( Ax \leq b \) 
\( x \geq 0 \)

**minimize** \( -c^T x \) 
subject to \( Ax - b \leq 0 \) 
\( -x \leq 0 \)

### Recall

Our Lagrange dual function for the above LP (to the right), defined over the domain \( A^T \lambda_1 - c - \lambda_2 = 0 \).

\[
g(\lambda) = -\lambda_1^T b
\]

The Lagrange dual problem can then be written as

**minimize** \( y^T b \) 
subject to \( A^T y \geq c \) 
\( y \geq 0 \)

**maximize** \( -\lambda_1^T b \) 
subject to \( A^T \lambda_1 - c - \lambda_2 = 0 \) 
\( A^T \lambda_1 \geq c \) 
\( \lambda \geq 0 \)
minimize $c^T x$
subject to $Ax = b$
$x \in K$

$x \in K$ can equivalently be written as $z^T x \leq 0, \forall z \in K^\circ$

$L(x, \lambda, \nu) = c^T x + \nu^T (Ax - b) + \sum_{z \in K^\circ} \lambda_z \cdot z^T x$

$$= (c - A^T \nu + \sum_{z \in K^\circ} \lambda_z \cdot z)^T x + \nu^T b$$
Another Example: Conic Optimization Problem

minimize \[ c^T x \]
subject to \[ Ax = b \]
\[ x \in K \]

- \( x \in K \) can equivalently be written as \( z^T x \leq 0, \forall z \in K^\circ \)

\[
L(x, \lambda, \nu) = c^T x + \nu^T (Ax - b) + \sum_{z \in K^\circ} \lambda_z \cdot z^T x
\]

\[
= (c - A^T \nu + \sum_{z \in K^\circ} \lambda_z \cdot z)^T x + \nu^T b
\]

- Can think of \( \lambda \succeq 0 \) as choosing some \( s \in K^\circ \)

\[
L(x, s, \nu) = (c - A^T \nu + s)^T x + \nu^T b
\]
Another Example: Conic Optimization Problem

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \in K
\end{align*}
\]

\[x \in K\] can equivalently be written as \(z^T x \leq 0, \forall z \in K^\circ\)

\[
L(x, \lambda, \nu) = c^T x + \nu^T (Ax - b) + \sum_{z \in K^\circ} \lambda_z \cdot z^T x
\]

\[
= (c - A^T \nu + \sum_{z \in K^\circ} \lambda_z \cdot z)^T x + \nu^T b
\]

Can think of \(\lambda \succeq 0\) as choosing some \(s \in K^\circ\)

\[
L(x, s, \nu) = (c - A^T \nu + s)^T x + \nu^T b
\]

Lagrange dual function \(g(s, \nu)\) is bounded when coefficient of \(x\) is zero, in which case it has value \(\nu^T b\)
Another Example: Conic Optimization Problem

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \in K \\
\text{maximize} & \quad \nu^T b \\
\text{subject to} & \quad A^T \nu - c \in K^o
\end{align*}
\]

- \( x \in K \) can equivalently be written as \( z^T x \leq 0, \forall z \in K^o \)

\[
L(x, \lambda, \nu) = c^T x + \nu^T (Ax - b) + \sum_{z \in K^o} \lambda_z \cdot z^T x
\]

\[
= (c - A^T \nu + \sum_{z \in K^o} \lambda_z \cdot z)^T x + \nu^T b
\]

- Can think of \( \lambda \succeq 0 \) as choosing some \( s \in K^o \)

\[
L(x, s, \nu) = (c - A^T \nu + s)^T x + \nu^T b
\]

- Lagrange dual function \( g(s, \nu) \) is bounded when coefficient of \( x \) is zero, in which case it has value \( \nu^T b \)
Outline

1. The Lagrange Dual Problem
2. Duality
Weak Duality

Primal Problem

\[ \min f_0(x) \]
\[ \text{s.t.} \]
\[ f_i(x) \leq 0, \quad \forall i = 1, \ldots, m. \]
\[ h_i(x) = 0, \quad \forall i = 1, \ldots, k. \]

Dual Problem

\[ \max g(\lambda, \nu) \]
\[ \text{s.t.} \]
\[ \lambda \geq 0 \]
Weak Duality

Primal Problem

\[ \min f_0(x) \]
\[ \text{s.t.} \]
\[ f_i(x) \leq 0, \quad \forall i = 1, \ldots, m. \]
\[ h_i(x) = 0, \quad \forall i = 1, \ldots, k. \]

Dual Problem

\[ \max g(\lambda, \nu) \]
\[ \text{s.t.} \]
\[ \lambda \succeq 0 \]

Weak Duality

\[ OPT(dual) \leq OPT(primal). \]

- We have already argued holds for every optimization problem
- **Duality Gap**: difference between optimal dual and primal values
Recall: Geometric Interpretation of Weak Duality

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_1(x) \leq 0
\end{align*}
\]

- Let \( G \) be attainable constraint/objective function value tuples
  - i.e. \((u, t) \in G\) if there is an \( x \) such that \( f_1(x) = u \) and \( f_0(x) = t \)
- \( p^* = \inf \{ t : (u, t) \in G, u \leq 0 \} \)
- \( g(\lambda) = \inf \{ \lambda u + t : (u, t) \in G \} \)
Recall: Geometric Interpretation of Weak Duality

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_1(x) \leq 0
\end{align*}
\]

- Let \( \mathcal{G} \) be attainable constraint/objective function value tuples
  - i.e. \((u, t) \in \mathcal{G}\) if there is an \( x \) such that \( f_1(x) = u \) and \( f_0(x) = t \)
  - \( p^* = \inf \{ t : (u, t) \in \mathcal{G}, u \leq 0 \} \)
  - \( g(\lambda) = \inf \{ \lambda u + t : (u, t) \in \mathcal{G} \} \)

**Fact**

The equation \( \lambda u + t = g(\lambda) \) defines a supporting hyperplane to \( \mathcal{G} \), intersecting \( t \) axis at \( g(\lambda) \leq p^* \).
We say strong duality holds if $OPT(dual) = OPT(primal)$.

- Equivalently: there exists a setting of Lagrange multipliers so that $g(\lambda, \nu)$ gives a tight lowerbound on primal optimal value.
- In general, does not hold for non-convex optimization problems.
- Usually, but not always, holds for convex optimization problems.
  - Mild assumptions, such as Slater's condition, needed.
Geometric Proof of Strong Duality

minimize \( f_0(x) \)
subject to \( f_1(x) \leq 0 \)

Let \( A \) be everything northeast (i.e. “worse”) than \( G \)

i.e. \( (u, t) \in A \) if there is an \( x \) such that \( f_1(x) \leq u \) and \( f_0(x) \leq t \)

\[ p^* = \inf \{ t : (0, t) \in A \} \]

\[ g(\lambda) = \inf \{ \lambda u + t : (u, t) \in A \} \]
Geometric Proof of Strong Duality

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_1(x) \leq 0
\end{align*}
\]

Let \( \mathcal{A} \) be everything northeast (i.e. “worse”) than \( \mathcal{G} \)

- i.e. \((u, t) \in \mathcal{A}\) if there is an \(x\) such that \(f_1(x) \leq u\) and \(f_0(x) \leq t\)

\[p^* = \inf \{t : (0, t) \in \mathcal{A}\}\]

\[g(\lambda) = \inf \{\lambda u + t : (u, t) \in \mathcal{A}\}\]

**Fact**

The equation \(\lambda u + t = g(\lambda)\) defines a supporting hyperplane to \(\mathcal{G}\), intersecting \(t\) axis at \(g(\lambda) \leq p^*\).
Geometric Proof of Strong Duality

minimize \( f_0(x) \)
subject to \( f_1(x) \leq 0 \)

**Fact**

When \( f_0 \) and \( f_1 \) are convex, \( A \) is convex.
Geometric Proof of Strong Duality

minimize \( f_0(x) \)
subject to \( f_1(x) \leq 0 \)

**Fact**

When \( f_0 \) and \( f_1 \) are convex, \( A \) is convex.

**Proof**

- Assume \((u, t)\) and \((u', t')\) are in \( A \)
minimize \( f_0(x) \)
subject to \( f_1(x) \leq 0 \)

**Fact**

When \( f_0 \) and \( f_1 \) are convex, \( \mathcal{A} \) is convex.

**Proof**

- Assume \((u, t)\) and \((u', t')\) are in \( \mathcal{A} \)
- \( \exists x, x' \) with \((f_1(x), f_0(x)) \leq (u, t)\) and \((f_1(x'), f_0(x')) \leq (u', t')\).
Geometric Proof of Strong Duality

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_1(x) \leq 0
\end{align*}
\]

**Fact**

When \( f_0 \) and \( f_1 \) are convex, \( A \) is convex.

**Proof**

- Assume \((u, t)\) and \((u', t')\) are in \( A \)
- \( \exists x, x' \) with \((f_1(x), f_0(x)) \leq (u, t)\) and \((f_1(x'), f_0(x')) \leq (u', t')\).
- By Jensen’s inequality \((f_1(\frac{x+x'}{2}), f_0(\frac{x+x'}{2})) \leq (\frac{u+u'}{2}, \frac{t+t'}{2})\)
Geometric Proof of Strong Duality

minimize \( f_0(x) \)
subject to \( f_1(x) \leq 0 \)

Fact
When \( f_0 \) and \( f_1 \) are convex, \( \mathcal{A} \) is convex.

Proof
- Assume \((u, t)\) and \((u', t')\) are in \( \mathcal{A} \)
- \( \exists x, x' \) with \((f_1(x), f_0(x)) \leq (u, t) \) and \((f_1(x'), f_0(x')) \leq (u', t') \).
- By Jensen’s inequality \((f_1(\frac{x+x'}{2}), f_0(\frac{x+x'}{2})) \leq (\frac{u+u'}{2}, \frac{t+t'}{2}) \)
- Therefore, midpoint of \((u, t)\) and \((u', t')\) also in \( \mathcal{A} \).
minimize \[ f_0(x) \]
subject to \[ f_1(x) \leq 0 \]

**Theorem (Informal)**

There is a choice of \( \lambda \) so that \( g(\lambda) = p^* \). Therefore, strong duality holds.
minimize $f_0(x)$ 
subject to $f_1(x) \leq 0$

**Theorem (Informal)**

There is a choice of $\lambda$ so that $g(\lambda) = p^*$. Therefore, strong duality holds.

**Proof**

- Recall $(0, p^*) \in A$
- By the supporting hyperplane theorem, there is a supporting hyperplane to $A$ at $(0, p^*)$
- Direction of the supporting hyperplane gives us an appropriate $\lambda$
minimize \( f_0(x) \)
subject to \( f_1(x) \leq 0 \)

In our proof, we ignored a technicality that can prevent strong duality from holding.
minimize \( f_0(x) \)
subject to \( f_1(x) \leq 0 \)

In our proof, we ignored a technicality that can prevent strong duality from holding.

If our supporting hyperplane \( H \) at \((0, p^*)\) is vertical, then no finite \( \lambda \) exists such that \((\lambda, 1)\) is normal to \( H \).
In our proof, we ignored a technicality that can prevent strong duality from holding.

If our supporting hyperplane $H$ at $(0, p^*)$ is vertical, then no finite $\lambda$ exists such that $(\lambda, 1)$ is normal to $H$.

Somewhat counterintuitively, this can happen even in simple convex optimization problems (though it's somewhat rare in practice).
minimize \ e^{-x} \\
subject to \ \ \ \ \ \ \ \ \ \ \ \frac{x^2}{y} \leq 0 \\
\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ y \geq 1 \\

- Problem is convex, with feasible region given by \ x = 0 \ and \ y \geq 1 \\
- Optimal value is 1, at \ x = 0 \ and \ y = 1
Violation of Strong Duality

minimize \( e^{-x} \)
subject to \( \frac{x^2}{y} \leq 0 \)
\( y \geq 1 \)

- Problem is convex, with feasible region given by \( x = 0 \) and \( y \geq 1 \)
- Optimal value is 1, at \( x = 0 \) and \( y = 1 \)
- Consider \( \mathcal{A} \) restricted to the objective and the first constraint
Violation of Strong Duality

minimize \( e^{-x} \)
subject to \( \frac{x^2}{y} \leq 0 \)
\( y \geq 1 \)

- Problem is convex, with feasible region given by \( x = 0 \) and \( y \geq 1 \)
- Optimal value is 1, at \( x = 0 \) and \( y = 1 \)
- Consider \( \mathcal{A} \) restricted to the objective and the first constraint
  \( \mathcal{A} = \mathbb{R}^2_{++} \cup \{0\} \times [1, \infty] \)
- Therefore, any supporting hyperplane to \( \mathcal{A} \) at \( (0, 1) \) must be vertical.
Slater’s Condition

There exists a point $x \in \mathcal{D}$ where all inequality constraints are strictly satisfied (i.e. $f_i(x) < 0$). I.e. the optimization problem is strictly feasible.

- A sufficient condition for strong duality.
- Forces supporting hyperplane to be non-vertical.
Slater’s Condition

There exists a point \( x \in D \) where all inequality constraints are strictly satisfied (i.e. \( f_i(x) < 0 \)). I.e. the optimization problem is **strictly feasible**.

- A sufficient condition for strong duality.
- Forces supporting hyperplane to be non-vertical
- Can be weakened to requiring strict feasibility only of non-affine constraints