Homework #1 Solutions
CS599 Fall 2012

November 25, 2012

General Instructions  The following problems are meant to be challenging. Feel free to discuss with fellow students, though please write up your solutions independently and acknowledge everyone you discussed the homework with on your writeup. Additionally, please provide mathematical proofs of all claims you make in your solutions.

Problem 1. Nash Equilibria in a Location Game.
Consider a walking trail along a stretch of beach, which we model as the interval $T = [0, 1]$. There are $k$ ice-cream vendors, each looking to set up an icecream stand at some point along the trail, with the goal of attracting as many customers as possible. We assume that pedestrians out for a walk are uniformly distributed along the trail, and when faced with a craving for some icecream head to the nearest icecream stand, so that the utility of an icecream stand $i$ is the measure of the set of points on the trail closer to $i$ than to any of $i$’s competitors. When two or more icecream stands are located at the same point $p$, they evenly split the customers closer to $p$ than any of the other locations of icecream stands.

Formally, each icecream vendor $i \in \{1, \ldots, k\}$ chooses a location $x(i) \in [0, 1]$ — i.e. his space of actions is $[0, 1]$. Given $x \in [0, 1]^k$ and $i \in \{1, \ldots, k\}$, let $[l_x(i), r_x(i)]$ be the line segment of points in $T$ no further from from $x(i)$ than from any other $x(j)$. Moreover, let $c_x(i)$ be the number of vendors co-located with $i$ — i.e. the number of vendors $j$ with $x(j) = x(i)$ (including $i$ himself). The utility of player $i$ is then defined as

$$u_i(x) = (r_x(i) - l_x(i))/c_x(i)$$

Clearly, this defines a game of complete information, where each player’s strategy set is $[0, 1]$. Answer the following:

a. (3 points). What is the set of pure Nash equilibria when $k = 2$?

Solution. There is one equilibrium at $(0.5, 0.5)$. It is easy to verify that $(0.5, 0.5)$ is an equilibrium. To see that it is unique, observe that no strategy profile in which the players play different strategies is an equilibrium, since a player would always gain by moving towards the other player. Moreover, any strategy profile $(x, x)$ with $x \neq 0.5$ is not an equilibrium: without loss of generality, assume $x < 0.5$, and observe that a player would gain by moving to the right.
b. (3 points). What is the set of pure Nash equilibria for \( k = 3 \)?

**Solution.** There are no equilibria. First, note that any strategy profile \((x, x, x)\) is not an equilibrium, as any player could move either to the right (when \( x \leq 0.5 \)) or left (when \( x \geq 0.5 \)) and improve their payoff. Second, note that in any equilibrium, the leftmost and rightmost player must not be unique (i.e. must share a location). To see that an action profile with a unique leftmost player (similarly, rightmost) is not an equilibrium, observe that the leftmost player may move to the right and increase his payoff. We conclude that there are no equilibria.

Now, consider a different trail \( T \) which goes around a circular lake — i.e. \( T \) is the circle in 2 dimensional space with circumference 1. The distance between two points \( p_1 \) and \( p_2 \) on \( T \) is naturally defined as the length of the shorter of the two arcs defined by \( p_1 \) and \( p_2 \). Define utilities as before: given a set of locations along \( T \) for each of the vendors, the utility of vendor \( i \) is the length of the arc of points on \( T \) which are no further from \( i \) than from any of \( i \)'s competitors, divided by the number of vendors co-located with \( i \). Answer the following:

c. (3 points). What is the set of pure Nash equilibria when \( k = 2 \)?

**Solution.** Every pure strategy profile is an equilibrium where each player has payoff 0.5.

d. (3 points). What is the set of pure Nash equilibria when \( k \geq 3 \)?

**Solution.** This is more complicated than I thought, due to co-located players. Everyone gets the 3 points for free.

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**Problem 2. Zero-sum Games and Linear Programming Review.**

A game in normal form is said to be a zero-sum game if, for every action profile of the players, the sum of players’ utilities is 0. Two-player zero-sum games exhibit specialized structure that renders them central to much of computer science and mathematical optimization. Specifically, zero-sum games are intimately related to linear programming, which is arguably the cornerstone of mathematical and combinatorial optimization. A canonical example of two-player zero-sum games is rock-paper-scissors.

Since the payoff of the second player is simply the negation of the payoff of the first player, we represent a two-player zero-sum game simply by a matrix \( A \in \mathbb{R}^{n \times m} \), where \( A_{ij} \) is the payoff of player 1 when player 1 plays strategy \( i \) and player 2 plays strategy \( j \). Let \( \Delta_k = \left\{ p \in \mathbb{R}_+^k : \sum_{i=1}^k p_i = 1 \right\} \) denote the simplex of dimension \( k \), let \( A_j \) denote the \( j \)'th column of \( A \), and let \( \overline{A}_i \) denote the \( i \)'th row of \( A \). Define the following two quantities:

\[
maxmin(A) = \max_{p \in \Delta_n} \min_{j=1}^m p^T A_j
\]

\[
minmax(A) = \min_{q \in \Delta_m} \max_{i=1}^n \overline{A}_i q
\]
Observe that $\text{maxmin}(A)$ is player 1's maximum, over all his mixed strategies $p$, of his utility when he plays $p$ and player 2 plays a best response to $p$. The $p$ achieving the maximum in this definition is referred to as player 1’s maxmin strategy. Similarly, $\text{minmax}(A)$ is player 2’s minimum, over all his mixed strategies $q$, of his loss (i.e. negation of his utility) when he plays $q$ and player 1 best responds to $q$. The $q$ achieving this minimum is referred to as player 2’s maxmin strategy.

**Groundwork.** Review linear programming and linear programming duality. I recommend Luca Trevisan’s notes here (chapters 5 and 6), though there are many good treatments of LP both in course notes and in textbooks. If you are confused, or want pointers to additional resources, come see me or write me.

a. Prove that, for every finite matrix $A$, $\text{maxmin}(A) = \text{minmax}(A)$. This is known as the minimax theorem, and the quantity $\text{maxmin}(A) = \text{minmax}(A)$ is referred to as the value of the game. You will need to invoke linear programming duality. (Do not hand in)

b. Prove that both players playing their maxmin strategies is a mixed Nash equilibrium of the zero-sum game. (Do not hand in)

c. **(10 points).** Consider the following 2-player, zero-sum game, defined on an undirected graph $G = (V, E)$ with a designated source node $s$, and target node $t$. The first player is the attacker, and his actions are the set of $s-t$ paths in $G$. Another player is the defender, and his actions are the set of edges in $G$. When the attacker plays a path $p$ and the defender plays an edge $e$, the utility of the attacker is 1 if $e \notin p$ and 0 if $e \in p$. Find the value of the game. You may express the value of the game in terms of the size of the minimum $s-t$ cut in $G$. (Hint: Invoke the minimax theorem and Menger’s theorem (google it)).

**Solution.** The value of the game is $1 - 1/C$, where $C$ is the size of the minimum $s-t$ cut in $G$. To show that the value of the game is at most $1 - 1/C$, it suffices by the minimax theorem to exhibit a mixed strategy for the defender for which no response by the attacker guarantees attacker payoff more than $1 - 1/C$. Indeed, consider the strategy of the defender in which he plays an edge $e$ from the minimum $s-t$ cut in $G$, chosen uniformly at random. Since every $s-t$ path passes through the min cut, any fixed such path $P$ contains $e$ with probability at least $1/C$. Therefore, no response by the attacker guarantees payoff more than $1 - 1/C$.

To show that the value of the game is at least $1 - 1/C$, it suffices by the minimax theorem to show a mixed strategy for the attacker which guarantees attacker payoff at least $1 - 1/C$ regardless of the defender’s response. In other words, it suffices to exhibit a distribution over $s-t$ paths which does not include any individual edge with probability more than $1/C$. By Menger’s theorem, there exists a set of $C$ edge-disjoint $s-t$ paths in $G$. Simply choosing one of these paths uniformly at random gives such a distribution.

**Problem 3. Existence of Bayes-Nash Equilibrium. (12 points).**

Prove that every finite Bayesian game of incomplete information, under the common prior assumption, admits a (mixed) Bayes-Nash equilibrium. (Hint: invoke Nash’s theorem, which states that every finite game of complete information admits a mixed Nash equilibrium)
\textbf{Solution.} Let $G = (N, A, T, u, D)$ be a game of incomplete information, where $N = \{1, \ldots, n\}$ is the set of players, $A_i$ is the set of actions of player $i$, $T_i$ is the set of types of player $i$, $u_i : T_i \times A \to \mathbb{R}$ is the utility function of player $i$, and $D$ is the common prior over $T = T_1 \times \ldots \times T_n$. We now define a game of complete information $G' = (N, A', u')$ and show that every equilibrium of the complete information game can be mapped to an equilibrium of the incomplete information game.

Let the set $A'_i$ of actions of player $i$ in the complete information game be his set of pure strategies $A^T_i$ in the incomplete information game. For an action profile $s \in A'$ of the complete information game (or equivalently, strategy profile in the incomplete information game), let $u'_i(s)$ be the expected utility of player $i$ in the incomplete information game over draws of all players’ types, including his own, when players play the strategy profile $s$. Specifically

$$u'_i(s) = \mathbb{E}_{t \sim D} u_i(t, s(t)).$$

Observe that the complete information game $G'$ takes an \textit{ex-ante} view of $G$; specifically, a player must choose his map from types to actions before learning of his own type or the types of others. This is in contrast to the standard \textit{interim} view of a game of incomplete information, where a player chooses his action after learning his own type, but before learning the types of others. Fortunately, however, the two perspectives are identical: a (possibly random) strategy $s_i : T_i \to A_i$ of player $i$ is an ex-ante best response (i.e. in expectation over all player types) if and only if $s_i(t_i)$ is an interim best response for every $t_i \in T_i$, in expectation over types of players other than $i$. Therefore, a mixed Nash equilibrium in the ex-ante game $G'$, guaranteed to exist by Nash’s theorem, is also a mixed Bayes-Nash equilibrium in the interim game $G$.

We now make this intuition more formal. Let $s_1, \ldots, s_n$ be a Nash equilibrium in the ex-ante game $G'$ — for a mixed equilibrium, think of $s_1, \ldots, s_n$ as independent random variables. We will show that $s_1, \ldots, s_n$ is also a mixed Bayes-Nash equilibrium in $G$. For each player $i$ and alternative strategy $s'_i$, we have the guarantee that:

$$\mathbb{E}_s[u'_i(s)] \geq \mathbb{E}_s u'_i(s'_i, s_{-i}).$$

Re-writing using the definition of $u'$, we have

$$\mathbb{E}_s \mathbb{E}_{t \sim D} u_i(t, s(t)) \geq \mathbb{E}_s \mathbb{E}_{t \sim D} u_i(t, (s'_i(t), s_{-i}(t))).$$

Since we are quantifying over all functions $s'_i : T_i \to A_i$, this is equivalent to the inequality holding for every type $t_i$ of player $i$, and every alternative action $a_i$ to $s_i(t_i)$; formally

$$\forall t_i \in T_i \forall a_i \in A_i \mathbb{E}_{t_{-i} \sim D|t_i} \mathbb{E}_{s \sim D} u_i(t_i, s(t)) \geq \mathbb{E}_{t_{-i} \sim D} \mathbb{E}_{s \sim D} u_i(t_i, (a_i, s_{-i}(t))).$$

Therefore, $s$ is a mixed Bayes-Nash equilibrium of the incomplete information game. \hfill \QED

\textbf{Problem 4. Auctions in More General Environments.}

Recall that the Vickrey auction is a dominant-strategy truthful mechanism which sells a single item to the player who values it most. Now, we will extend the Vickrey auction to settings where
the seller has multiple copies of the item for sale. As in single-item allocation, we assume players exhibit quasi-linear utility in all these examples — i.e. a player’s utility for an allocation and payment is his value for the allocation less his payment.

a. (3 points). Consider an auctioneer with \( k \) identical items for sale, and \( n \) players interested in at most one item each. As in the single-item allocation problem, each player \( i \)’s type is a real number \( v_i \), encoding his value for receiving at least 1 item. Assume that a player exhibits unit demand, meaning that his value for more than one item is the same as his value for a single item. Design a dominant-strategy truthful mechanism that maximizes social welfare, defined as the sum over all players of their value for the items they receive. Your mechanism may charge payments, as in the Vickrey auction. (Hint: the mechanism will be a natural generalization of the Vickrey auction).

Solution. The auction solicits bids, awards an item to the \( k \) highest bidders, and charges each the \( k+1 \)'st highest bid. This is dominant-strategy truthful by the observation that the auction is equivalent to that in which each player \( i \) faces a take-it-or-leave-it offer, equal to the \( k \)'th highest bid among players other than \( i \).

b. (Adapted from problem 2.3 in Hartline). (8 points). Now, consider a generalization of the \( k \)-item allocation problem in (a) above, faced in online advertising. Adwords is a Google product in which the search engine sells at auction advertisements that appear along side search results on the search results page.

Consider the following position auction environment which provides a simplified model of Adwords. There are \( m \) advertisement slots that appear along side search results and \( n \) advertisers (the players, or bidders). Each advertiser’s type is his value \( v_i \) for a click on his advertisement. Slot \( j \) has click through rate \( w_j \), meaning, if an advertiser is assigned slot \( j \) the advertiser will receive a click with probability \( w_j \). Therefore, player \( i \)’s value for being placed in slot \( j \) is \( v_j w_j \). Each advertiser can be assigned at most one slot and each slot can be assignment at most one advertiser. You may assume, for convenience, that \( w_1 \geq w_2 \geq \ldots \geq w_m \).

Design a dominant-strategy truthful mechanism that maximizes social welfare, defined as the sum over all players of their value per click multiplied by the click-through rate of the slot to which they are assigned, if any (else 0). As before, you may charge payments. (Hint: you can map the problem of welfare maximization in position auctions to \( m \) separate problems of welfare-maximization in \( k \)-item auctions (see subproblem a)).

Solution. Assume without loss of generality that \( m = n \), either by adding dummy slots of click-through rate zero, or by discarding slots. The auction solicits bids, sorts the players in decreasing order of bids \( b_i \), and assigns the \( i \)’th player to the \( i \)’th slot. Player \( i \) is then charged \( \sum_{j=n-1}^{i}(w_j - w_{j+1})b_{j+1} \). This clearly maximizes social welfare. To see that it is truthful, observe that it is equivalent to running a separate multi-item auction for each “incremental” increase in click through rate \( \Delta w_j = w_j - w_{j+1} \). The auction for the \( \Delta w_j \) increment is simply a \( j \)-item auction where each player \( i \)’s value is interpreted as \( b_i \Delta w_j \). Each of these incremental auctions is truthful, and player utility is additive over these auctions, therefore the resulting “laddered auction” is truthful as well.
c. (5 points). Now, consider different generalization of the $k$-item auction. A bipartite graph encodes sets of players who may jointly receive an item, as follows. Specifically, the bipartite graph $G$ has players on the left hand side, and $k$ non-identical items on the right hand side. The mechanism may assign item $j$ to player $i$ only if there is an edge between player $i$ and item $j$ in $G$. A player has the same value $v_i$, encoded by his type, for any item which may be assigned to him (as specified by $G$). Moreover, all players exhibit unit demand, as in subproblem (b).

Design a dominant-strategy truthful mechanism that maximizes social welfare, defined as the sum, over all players $i$ who receive an item, of the player’s value $v_i$ for an item. (Note: The $k$ item auction of subproblem (a) is encoded by the complete bipartite graph.)

Solution. Whereas a solution to this question follows from the VCG mechanism, we instead derive an auction from “first principles.” The allocation rule of our auction solicits bids, labels edges in the bipartite graph with the bid of the associated player, and computes a maximum-weight matching $M^*$ in the resulting weighted bipartite graph. It is easy to see that this allocation rule is monotone. For convenience, we assume that the maximum weight matching is unique, which can be enforced by slight perturbation of the weights.

Now, we show how to compute the threshold payment rule for a player $i$. Let $M^*$ be the set of players served in the maximum matching, and let $M^*_{-i}$ denote the set of players served in the maximum matching computed after excluding player $i$. We will show that either $M^*_{-i} \subseteq M^*$, in which case we charge player $i$ nothing, or else there is a “replacement player” $r(i)$, independent of player $i$’s bid, such that $M^* = M^*_{-i} \cup \{i\} \setminus \{r(i)\}$. From that it follows that $i \in M^*$ if and only if $b_i \geq b_{r(i)}$, and therefore charging player $i$ then amount $b_{r(i)}$ when included in $M^*$ results in a truthful mechanism.

To show the existence of the replacement player $r(i)$ as claimed, we make an appeal to the theory of augmenting paths (aka alternating paths) in bipartite matching. It is known that the symmetric difference of any two bipartite matchings $M_1$ and $M_2$ can be partitioned into node-disjoint cycles and paths, each of whose edges alternates in membership between $M_1$ and $M_2$. Therefore, $M_1$ can be transformed into $M_2$ by a sequence of “edge swaps” along each cycle/path, where the cycles/paths can be processed in an arbitrary order. Consider $M_1 = M^*_{-i}$ and $M_2 = M^*$. We claim that the symmetric difference includes at most a single path, which directly implies the existence of the replacement player $r(i)$ as claimed. Indeed, if the symmetric difference included more than a single path or cycle, one of those must not involve $i$ – call it $P$. Since both $M^*_{-i} \oplus P$ and $M^* \oplus P$ are matchings, either $M^*_{-i}$ or $M^*$ is not optimal, a contradiction.