Logic 450 Midterm
Due 12pm October 27

Explain your reasoning clearly for each question. You can use any of the facts we've already proved in class without further justification, but be explicit about which facts you are using.

You may drop one question: turn in answers to any seven out of the eight questions.

1. Let A and B be any sets. Suppose $A \cong A'$ (that is, A and A' are in one-to-one correspondence). Show the following.

   (a) If there is a one-to-one function $f : A \to B$, then there is a one-to-one function $f' : A' \to B$

   (b) If A is countable, then A' is countable.

2. Let A be a set.

   If B is any subset of A, the **characteristic function** of B is the function $\mathsf{char}\, B : A \to \mathbb{2}$ such that

$$(\mathsf{char}\, B)a = \begin{cases} \top & \text{if } a \in B \\ \bot & \text{otherwise} \end{cases} \tag{1}$$

   (That is, $\mathsf{char}\, B$ is True for each element of B, and False for each non-element of B.)

   For any function $f : A \to \mathbb{2}$, the **kernel** of f is the set

$$\ker f = \{a \in A \mid fa = \top\} \tag{2}$$

   (That is, $\ker f$ is the set of elements for which the value of f is True.)

   (a) For any subset B of A, the kernel of the characteristic function of B is B. That is,

$$\ker(\mathsf{char}\, B) = B \tag{3}$$

(b) For any function $f : A \to \mathbb{2}$, the characteristic function of the kernel of f is f. That is,

$$\mathsf{char}(\ker f) = f \tag{4}$$

(c) Conclude that the set of functions $A \to \mathbb{2}$ is in one-to-one correspondence with the power set of A. That is,

$$\mathbb{2}^A \cong PA \tag{5}$$

3. The **length** of a sentence of $\mathsf{Prop}\, S$ is, intuitively, the total number of sentence letters or connectives that appears in it. For example:

   - The length of $p \wedge q$ is 3.
   - The length of $p \wedge (p \to \bot)$ is 5.
   - The length of $(\bot \to (\bot \to \bot))$ is 5.

   (a) Give a recursive definition of the length of a sentence.

   (b) Prove by induction that, for any $A \in \mathsf{Prop}\, 1$, and any $B \in \mathsf{Prop}\, 2$, the length of the substitution instance $A[B]$ is at least the length of A.

   (c) Give an example of some $A \in \mathsf{Prop}\, 1$ and some $B \in \mathsf{Prop}\, 2$ such that the length of $A[B]$ is less than the length of B.

4. Prove from the definitions that for any propositional sentences A, B, C,

$$A, B \vDash C \quad \text{iff} \quad (A \wedge B) \to C \text{ is valid} \tag{6}$$

5. Let X be a consistent and negation-complete set of sentences. Suppose that $A, B \vdash C$. Show that C is an element of X or $(A \wedge B) \to \bot$ is an element of X.

6. For any set of sentence letters $P \subseteq S$, let M_P be the truth-assignment that assigns True to every sentence letter in P, and False to every sentence letter not in P.

$$M_P(s) = \begin{cases} \top & \text{if } s \in P \\ \bot & \text{otherwise} \end{cases} \tag{7}$$

   (In other words, M_P is the characteristic function $\mathsf{char}\, P$.)

Let X be a set of sentences in $\mathsf{Prop}\, S$. Suppose that for every *finite* set of sentence letters $P \subseteq S$, every sentence in X is true in M_P. Show that every sentence in X is true in M_S (the truth-assignment which assigns True to *every* sentence letter).

Hint. Use Compactness.

7. Suppose S and T are structures for a signature with a single one-place function symbol f. Suppose that $S \cong T$ (that is, S and T are isomorphic). Show that if f_T is a one-to-one function, then f_S is also a one-to-one function.

8. Let L be a countable signature, and let S be an L-structure with a countably infinite domain. Call a function $f : D_S \to D_S$ **expressible** iff there is some term $a \in \mathsf{Term}_L\, 1$ such that $[\![a]\!]_S = f$. Prove that there exists some function $f : D_S \to D_S$ which is not expressible.