Logic 450 Final Exam
Due in my mailbox in the Mudd Hall main office (or by email) by **5pm December 17th.**

Explain your reasoning clearly for each question. You can use any of the facts we've already proved in class without further justification, but be explicit about which facts you are using. **You may drop two questions**: turn in answers to any 8 out of the 10 questions.

1. Let R be any set of pairs of elements of A. We say R is an **equivalence relation** iff for every $a, b, c \in A$,

$$\begin{aligned} (a, a) \in R & \quad \text{(Reflexivity)} \\ \text{If } (a, b) \in R \text{ then } (b, a) \in R & \quad \text{(Symmetry)} \\ \text{If } (a, b) \in R \text{ and } (b, c) \in R \text{ then } (a, c) \in R & \quad \text{(Transitivity)} \end{aligned} \tag{1}$$

   If R is an equivalence relation, then the **equivalence class** of a is the set $\{b \in A \mid (a, b) \in R\}$.

   (a) Show that if R is an equivalence relation, then for any elements a and b in A, $(a, b) \in R$ iff a and b have the same equivalence class.

   (b) Show that R is an equivalence relation if and only if there is some function f with domain A such that, for every $a, b \in A$,

$$(a, b) \in R \quad \text{iff} \quad f(a) = f(b) \tag{2}$$

2. Say whether each of the following statements is **true** or **false**. Briefly explain your answers.

   (a) For any countable set A, the set of all pairs of elements of A is countable.

   (b) For any countable set A, the set of all subsets of A is countable.

   (c) For any countable set A, the set of all *finite* subsets of A is countable.

   (d) For any countable set A and any equivalence relation R on A, the set of all equivalence classes of R is countable.

   (e) For any countable set A, if there is a one-to-one function from A to B, then B is countable.

3. Say a two-place function $g : D \times D \to D$ is **transitive** iff for any a and b in D, there is some c in D such that $g(a, c) = b$. Suppose that f is a two-place function symbol in the signature L, and suppose that S and T are isomorphic L-structures. Show that if f_S (the interpretation of f in S) is transitive, then so is f_T (the interpretation of f in T).

4. Let X be a consistent, negation-complete, and witness-complete set of sentences. Show that for any one-variable formula $A[x]$, if $\exists x\, A[x]$ is an element of X, then for some term a, $A[a]$ is an element of X.

5. Let L be any first-order language. Let X be a set of L-sentences, let $A[x]$ be a formula of one variable in L, and let B and C be L-sentences. Prove the following. (Feel free to use Soundness and Completeness.)

   (a) $B \to \neg B \underset{X}{\equiv} \neg B$

   (b) $X, B \vdash C$ iff $X, \neg C \vdash \neg B$

   (c) If a is any term in L, then $A[a] \vDash \exists x\, A[x]$

   (d) Let a be any name which is *not* in L.

$$\text{If} \quad X, A[a] \underset{L+a}{\vDash} B \quad \text{then} \quad X, \exists x\, A[x] \underset{L}{\vDash} B \tag{3}$$

6. Consider a language L which includes some formula $A[x]$.

   (a) Explain why for any number n, there is a sentence B_n which is true in every L-structure in which at least n things satisfy $A[x]$, and which is true in no other L-structures. (It's not necessary to spell this out in a lot of detail.)

   (b) Call an L-structure "good" iff there are only finitely many things in its domain that satisfy $A[x]$; otherwise call it "bad". Suppose that for each n, the sentence B_n defined in part (a) is true in some good structure. (In other words, it is logically possible for there to be an arbitrarily large finite number of things that satisfy $A[x]$.) Show that there is *no* L-theory which is true in every good structure, but no bad structures.

   (In other words, this problem shows that "There are finitely many A's" is inexpressible in first-order logic.)

7. Suppose that X is a consistent, categorical theory. Show that there is some natural number n such that every model of X has exactly n elements.

(For this problem you may use the "upward" Löwenheim-Skolem theorem, which was proved informally in class but was not an exercise in the notes: for any theory X, if X has an infinite model, then X has an uncountable model.)

8. Let X be a theory that defines substitution. (This implies that X has quotation-terms that denote each sentence and one-variable formula in its language.) Suppose that X has an open formula $\mathcal{N}[x]$ of one variable. (Intuitively we'll understand $\mathcal{N}[x]$ as meaning "The sentence x is necessarily true".) Suppose also that the following two principles hold whenever the term q denotes the sentence A:

$$X \vdash \mathcal{N}[q] \to A \tag{4}$$

$$\text{If } X \vdash A \quad \text{then} \quad X \vdash \mathcal{N}[q] \tag{5}$$

(The first principle intuitively corresponds to the modal principle called "T": whatever is necessarily true, is actually the case. The second principle intuitively corresponds to the rule in modal logic called "necessitation": if you can prove a statement, then it is necessarily true.)

Show that X is inconsistent.

The last two problems use the following assumption and definition:

- **Assumption.** There is a quotation system for the language of arithmetic. The theory of arithmetic $\mathsf{Th}\,\mathbb{N}$ (that is, the set of all sentences in the language of arithmetic which are true in the standard model of arithmetic $\mathbb{N}$) defines substitution relationally.
- **Definition.** Suppose X is any set of sentences (in the language of arithmetic). If $A[x]$ is a one-variable formula in the language of arithmetic, we say $A[x]$ **defines** X (in arithmetic) iff, for every term q that denotes a sentence B,

$$B \in X \quad \text{iff} \quad \mathsf{Th}\,\mathbb{N} \vdash A[q] \tag{6}$$

We say X is **definable** (in arithmetic) iff there is some formula $A[x]$ that defines X. Otherwise X is **undefinable**.

9. Prove that there are uncountably many undefinable sets of sentences.

10. Show that $\mathsf{Th}\,\mathbb{N}$, the set of all truths in arithmetic, is undefinable.