Foundations of Arithmetic Exercises

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Week 4

We're not looking for completely formal proofs of the sort Frege carried out in the *Grundgesetze*. (We haven't discussed any formal system for reasoning with concepts.) Instead, the goal is to provide proofs that are as rigorous as you would normally do for a math class. You can take for granted facts about the logic of concepts. (For example, for any concepts F and G and object a, there is such a concept as $\lambda x.Fx \vee Gx \vee x = a$. More generally, you can *abstract* any well-formed description $\phi(x)$ to come up with a concept $\lambda x.\phi(x)$. But of course, don't take facts about arithmetic for granted!)

Definition 1. Let R be a relation-concept.

a. R is **one-to-one** iff $\forall xx'yy'(Rxy \wedge Rx'y' \to (x = x' \leftrightarrow y = y'))$.
b. R has **domain** F iff $\forall x(Fx \leftrightarrow \exists y\, Rxy)$
c. R has **range** G iff $\forall y(Gy \leftrightarrow \exists x\, Rxy)$
d. If F and G are concepts, $F \sim G$ iff there is some relation R which is one-to-one and has domain F and range G. (We also say F is **equinumerous** to G; in Austin's translation, "equal".)

Postulate (Hume's Principle). $\#F = \#G$ iff $F \sim G$.

This postulate builds in the assumption that *there is* such an object as $\#F$, for any concept F. It also builds in the assumption that there is just one such object (though we could also prove this from Definition 1).

Definition 2. n is a **number** iff $\exists F(n = \#F)$.

Definition 3. $0 = \#(\lambda x.\, x \neq x)$.

Exercise 1. $\#F = 0$ iff $\forall x\, \neg Fx$. (§75)

Example solution. Say a concept F is **empty** iff for no x, Fx. Note first that for any relation R, R has an empty domain iff R has an empty range, and in that case R is also automatically one-to-one. Then:

$\#F = 0$ iff $\#F = \#(\lambda x.\, x \neq x)$	(definition of 0)
iff $F \sim \lambda x.\, x \neq x$	(Hume's Principle)
iff some one-to-one R has domain F and range $\lambda x.\, x \neq x$	(definition of $\sim$)
iff some one-to-one R has domain F and an empty range	
iff F is empty.	

QED.

Definition 4. n **succeeds** m iff there is some concept F and some object a such that Fa, $n = \#F$, and $m = \#(\lambda x.\, x \neq a \wedge Fx)$.

Exercise 2. 0 has a successor. (I.e., something succeeds 0.) (§77)

Exercise 3. If $\#F$ succeeds 0 then there is some object a such that Fa and $\forall x(Fx \to x = a))$.

Exercise 4. If there is some object a such that Fa and $\forall x(Fx \to x = a)$, then $\#F$ succeeds 0.

Exercise 5. If n succeeds m and n' succeeds m then $n = n'$.

By Ex 2 and 5, it makes sense to talk about *the* successor of 0. So we can say:

Definition 5. 1 is the successor of 0.

Then Ex 3 and 4 together show that Frege's earlier definition of "The number of F is one" holds iff $\#F = 1$.

Exercise 6. If n succeeds m and n succeeds m' then $m = m'$.

From Ex 5 and 6 it follows that the relation-concept ($\lambda mn.\, n$ succeeds m) is one-to-one. (§78)

Exercise 7. If n is a number then either $n = 0$ or $\exists m(n$ succeeds $m)$.

Exercise 8. 0 does not succeed anything.

The main thing we still have to show that every number has a successor. We'll use the same basic idea as we used for proving the existence of 1: the successor of n is the number of the concept of being identical to 0, or 1, or 2, …, or n. The trick is to state this precisely—which is surprisingly tricky. To do this, we'll use the following definition.

Definition 6. The (**weak**) **ancestral** of R is the relation-concept

$$\lambda ba\, \forall F(Fa \wedge \forall xy(Ryx \wedge Fx \to Fy) \to Fb)$$

Denote this relation R^*.

This defines a concept slightly different from the one Frege defines in §79, "y follows in the R-series after x". Make sure you see how it is different. Also note the similarity of this definition to the principle of induction. Arguments which appeal to this definition usually have the same structure as standard inductive proofs.

Definition 7. $n \geq m$ iff n succeeds* m. That is, $n \geq m$ iff: for every concept F, if Fm, and for all x and y (if y succeeds x and Fx then Fy), then Fn.

Exercise 9. Suppose n succeeds b, m succeeds a, and $n \geq m$. Then $b \geq a$. (Hint: use the definition of $n \geq m$ with the concept $\lambda n.\, n$ succeeds some $x \geq a$.)

Definition 8. A number n is **finite** iff $n \geq 0$.

(Alternatively we could call finite numbers **natural numbers**.)

Exercise 10. No finite number is its own successor.

Exercise 11. Suppose k falls under the concept $\lambda k.\, \forall n(n \geq k \wedge k \geq n \to k = n)$. If m succeeds k, then m also falls under that concept.

Exercise 12. Suppose m is a finite number such that $m \geq n$ and $n \geq m$. Then $m = n$.

Exercise 13. Suppose m is a finite number, n succeeds m, and $m \geq k$. Then $k \neq n$.

Exercise 14. If n succeeds m and $n \geq k$, then $n = k$ or $m \geq k$.

Exercise 15. If n is a finite number, then $n = \#(\lambda k.\, n \neq k \wedge n \geq k)$. (§83)

Exercise 16. For any finite number n, $\#(\lambda k.\, n \geq k)$ succeeds n. Thus every finite number has a successor.

Now that we know each number has a unique successor (Ex. 5 and 16) we are also justified in writing Sn to denote the unique successor of n, as long as n is a finite number. And we can let $1 = S0$, $2 = S1$, etc. Note that the **principle of induction** immediately follows from Def 7 and 8:

Principle of Induction. For any concept F, if $F0$, and for any finite number n, if Fn then $F(Sn)$, then it follows that every finite number falls under F. In symbols: $\forall F(F0 \wedge \forall n(Fn \to F(Sn)) \to \forall n\, Fn)$.

The next thing we'll do is show that addition is well-defined. The idea is that addition obeys the following conditions:

i. $m + 0 = m$
ii. $m + Sn = S(m + n)$

The trick is to show that there really is some three-place relation that obeys these conditions. We'll do this by first defining certain *partial* sum-relations, which are only defined for finitely many different arguments. Then the *sum* relation will be the combination of all of these.

Definition 9. Let m and n be finite numbers. Call a two-place relation R a **pre-sum relation for** m **up to** n iff for all n' and k, $R(n', k)$ iff $n \geq n'$ and either (i) $n' = 0$ and $k = m$ or (ii) for some n'' and k', n' succeeds n'' and k succeeds k' and $R(n'', k')$.

Exercise 17. For any finite numbers m and n, there is a pre-sum relation for m up to n. (Hint: use induction on n.)

Definition 10. Say k is a sum of m and n iff for some R which is a pre-sum for m up to n, $R(m, n, k)$

Exercise 18. Any two finite numbers m and n have exactly one sum, which we can denote $m + n$.

Exercise 19. For all finite numbers m and n, $m + 0 = m$ and $m + Sn = S(m + n)$.

Exercise 20. $2 + 2 = 4$