Solutions: Recursion and Induction

Exercise 1(a)

Let $M = (T, T, \ldots, T)$. We’ll show by induction that for any sentence, if $A$ has no occurrences of $\bot$, then $⟦A⟧_M = T$. There are four cases to consider.

1. For any sentence letter $s$, $⟦s⟧_M = M(s) = T$.
2. For $\bot$, the conditional is trivially true.
3. If $B \land C$ contains no occurrences of $\bot$, then $B$ and $C$ each contain no occurrences of $\bot$. So, by the inductive hypothesis, $⟦B⟧_M = ⟦C⟧_M = T$. It follows that

$$⟦B \land C⟧_M = ⟦B⟧_M \land ⟦C⟧_M = T \land T = T$$

(1)

4. For the case $B \rightarrow C$, as in case 3 we can assume that $⟦B⟧_M = ⟦C⟧_M = T$. Then

$$⟦B \rightarrow C⟧_M = ⟦B⟧_M \rightarrow ⟦C⟧_M = T \rightarrow T = T$$

(2)

So we conclude by induction that for every sentence $A$, $⟦A⟧_M = T$. This means that $⟦A⟧$ is positive.

Exercise 1(b)

Consider the truth-function $\bot$ that takes every $M \in 2^S$ to $\bot$. By part (a), for any sentence $A$ that contains no $\bot$, then $⟦A⟧_M \neq \bot(M)$ when $M = (T, \ldots, T)$, and so $⟦A⟧ \neq \bot$.

Exercise 2

Let $B$ be any sentence, and let $M$ be any model. We’ll show by induction that for any sentence $A \in \text{Prop} 1$, if $A$ contains no conditionals, if $⟦A[B]⟧_M = T$, then $⟦B⟧_M = T$. There are four cases to consider.

1. For the sentence letter $p$, $⟦p⟧ = B$. So clearly if $⟦p[B]⟧_M = T$ then $⟦B⟧_M = T$.
2. $⟦\bot[B]⟧_M = ⟦\bot⟧_M = \bot$. This can’t be $T$, so our conclusion holds trivially in this case.
3. If a conjunction $A_1 \land A_2$ contains no conditionals, then neither does $A_1$ or $A_2$. So the inductive hypothesis says:

\[
\text{If } \llbracket A_1[B] \rrbracket_M = \top \text{ then } \llbracket B \rrbracket_M = \top \\
\text{If } \llbracket A_2[B] \rrbracket_M = \top \text{ then } \llbracket B \rrbracket_M = \top
\] (3)

Using the definitions of substitution and the interpretation function:

\[
\llbracket (A_1 \land A_2)[B] \rrbracket_M = \llbracket A_1[B] \land A_2[B] \rrbracket_M = \llbracket A_1[B] \rrbracket_M \land \llbracket A_2[B] \rrbracket_M
\] (4)

So if $\llbracket (A_1 \land A_2)[B] \rrbracket_M = \top$ then $\llbracket A_1[B] \rrbracket_M = \top$ and $\llbracket A_2[B] \rrbracket_M = \top$. Therefore, $\llbracket B \rrbracket_M = \top$, by the inductive hypothesis, which is what we needed to show.

4. The conditional $A_1 \rightarrow A_2$ obviously contains a conditional, so again our conclusion holds trivially.

It follows that for every model, if $\llbracket A[B] \rrbracket_M = \top$ then $\llbracket B \rrbracket_M = \top$. This means that $A[B] \models B$.

**Exercise 3(a)**

1. $\delta(s) = 0$ for any $s \in S$
2. $\delta(\bot) = 0$
3. $\delta(A \land B) = \delta(A) + \delta(B) + 1$
4. $\delta(A \rightarrow B) = \delta(A) + \delta(B)$

**Exercise 3(b)**

For the inductive proof there are four cases.

1. For a sentence letter $s \in S$, $\delta(s) = 0$, and since $\delta(s[B])$ is a natural number, $\delta(s[B]) \geq \delta(s)$.
2. $\delta(\bot) = 0$, so the same reasoning applies as case 1.
3. By the definitions of substitution and $c$,
\[
\epsilon((A_1 \rightarrow A_2)[B]) = \epsilon(A_1[B] \rightarrow A_2[B]) = \epsilon(A_1[B]) + \epsilon(A_2[B]) \tag{5}
\]
By the inductive hypothesis, $\epsilon(A_1[B]) \geq \epsilon(A_1)$ and $\epsilon(A_2[B]) \geq \epsilon(A_2)$. It follows that
\[
\epsilon(A_1[B]) + \epsilon(A_2[B]) \geq \epsilon(A_1) + \epsilon(A_2) = \epsilon(A_1 \rightarrow A_2) \tag{6}
\]
Putting these together, we see that
\[
\epsilon((A_1 \rightarrow A_2)[B]) \geq \epsilon(A_1 \rightarrow A_2) \tag{7}
\]

4. By the definitions of substitution and $c$,
\[
\epsilon((A_1 \land A_2)[B]) = \epsilon(A_1[B] \land A_2[B]) = \epsilon(A_1[B]) + \epsilon(A_2[B]) + 1 \tag{8}
\]
By the inductive hypothesis, $\epsilon(A_1[B]) \geq \epsilon(A_1)$ and $\epsilon(A_2[B]) \geq \epsilon(A_2)$. It follows that
\[
\epsilon(A_1[B]) + \epsilon(A_2[B]) + 1 \geq \epsilon(A_1) + \epsilon(A_2) + 1 = \epsilon(A_1 \land A_2) \tag{9}
\]
Putting these together,
\[
\epsilon((A_1 \land A_2)[B]) \geq \epsilon(A_1 \land A_2) \tag{10}
\]

**Exercise 3(c)**

There are four cases.

1. For a sentence letter $t \in S$,
\[
\#(s, t) = \begin{cases} 
1 & \text{if } s = t \\
0 & \text{otherwise}
\end{cases} \tag{11}
\]

2. $\#(s, \bot) = 0$

3. $\#(s, A \land B) = \#(s, A) + \#(s, B)$
4. \( \#(s, A \rightarrow B) = \#(s, A) + \#(s, B) \)

**Exercise 3(d)**

There are four cases to consider for an inductive proof:

1. By the definition of substitution, \( p[B] = B \). Also \( c(p) = 0 \), and \( \#(p, p) = 1 \). So:
   \[
   c(p[B]) = c(B) = 0 + c(B) \cdot 1 = c(p) + c(B) \cdot \#(p, p)
   \]  
   (12)

2. By the definitions, \( \bot[B] = \bot \), \( c(\bot) = 0 \), and \( \#(p, \bot) = 0 \). So:
   \[
   c(\bot[B]) = c(\bot) = 0 = 0 + c(B) \cdot 0 = c(\bot) + c(B) \cdot \#(p, \bot)
   \]  
   (13)

3. For a conjunction \( A_1 \land A_2 \), the inductive hypothesis tells us:
   \[
   c(A_1[B]) = c(A_1) + c(B) \cdot \#(p, A_1)
   c(A_2[B]) = c(A_2) + c(B) \cdot \#(p, A_2)
   \]  
   (14)

   By the definitions,
   \[
   c((A_1 \land A_2)[B]) = c(A_1[B] \land A_2[B]) = c(A_1[B]) + c(A_2[B]) + 1
   \]  
   (15)

   Applying the inductive hypothesis, this is equal to
   \[
   c(A_1) + c(B) \cdot \#(p, A_1) + c(A_2) + c(B) \cdot \#(p, A_2) + 1
   \]  
   (16)

   Rearranging terms, this is equal to
   \[
   (c(A_1) + c(A_2) + 1) + c(B) \cdot (\#(p, A_1) + \#(p, A_2))
   \]  
   (17)

   And by the definitions of \( c \) and \( \# \) this is the same as
   \[
   c(A_1 \land A_2) + c(B) \cdot \#(p, A_1 \land A_2)
   \]  
   (18)

4. Similar to case 3.
Exercise 4(a)

\[ \text{sl}(s) = \{ s \} \quad \text{for a sentence letter } s \in S \]
\[ \text{sl}(\bot) = \{ \} \]
\[ \text{sl}(A \land B) = \text{sl}(A) \cup \text{sl}(B) \]
\[ \text{sl}(A \rightarrow B) = \text{sl}(A) \cup \text{sl}(B) \]

(19)

Exercise 4(b)

We’ll show by induction that for every sentence \( A \), if \( M \) and \( M' \) have the same truth value for every sentence letter in \( A \), then \( \llbracket A \rrbracket_M = \llbracket A \rrbracket_{M'} \). There are four cases to consider!

1. For a sentence letter \( s \), since obviously \( s \in \text{sl}(s) = \{ s \} \),
   \[ \llbracket s \rrbracket_M = M(s) = M'(s) = \llbracket s \rrbracket_{M'} \]
   (20)

2. \( \llbracket \bot \rrbracket_M = \bot = \llbracket \bot \rrbracket_{M'} \)

3. Consider a conjunction \( A \land B \). If \( M(s) = M'(s) \) for every \( s \in \text{sl}(A \land B) = \text{sl}(A) \cup \text{sl}(B) \), it follows that \( M(s) = M'(s) \) for every \( s \in \text{sl}(A) \), and also \( M(s) = M'(s) \) for every \( s \in \text{sl}(B) \). So by the inductive hypothesis,
   \[ \llbracket A \rrbracket_M = \llbracket A \rrbracket_{M'} \]
   \[ \llbracket B \rrbracket_M = \llbracket B \rrbracket_{M'} \]
   (21)

   Then we can conclude:
   \[ \llbracket A \land B \rrbracket_M = \llbracket A \rrbracket_M \land \llbracket B \rrbracket_M = \llbracket A \rrbracket_{M'} \land \llbracket B \rrbracket_{M'} = \llbracket A \land B \rrbracket_{M'} \]
   (22)

4. Similar to case 3.

Exercise 4(c)

If neither \( A \equiv \bot \) nor \( B \equiv \top \), then there must be some model \( M \) such that \( \llbracket A \rrbracket_M = \top \), and there must be some model \( M' \) such that \( \llbracket B \rrbracket_{M'} \). In that case, define a new model \( M'' \).
\[ M^*(s) = \begin{cases} M(s) & \text{if } s \neq p \\ M'(s) & \text{otherwise} \end{cases} \quad (23) \]

Note that for every \( s \neq p \), \( M^*(s) = M(s) \). Since \( p \notin s A \), it follows that \( M(s) = M^*(s) \)
for every \( s \in s A \). So by part (b),

\[ \llbracket A \rrbracket_{M^*} = \llbracket A \rrbracket_M = \top \quad (24) \]

Similarly, since \( s A = \{ p \} \), and \( M'(p) = M^*(p) \), by part (b) again,

\[ \llbracket B \rrbracket_{M^*} = \llbracket B \rrbracket_{M'} = \bot \quad (25) \]

So \( M^* \) is a model in which \( A \) is true and \( B \) is false. So it is not the case that \( A \vDash B \).
Contrapositively, if \( A \vDash B \), then either \( A \vDash \bot \) or else \( B \).

Note the proof works with hardly any changes if instead we just assume:

\[ s A \cap s B = \{ \} \quad (26) \]