Solutions: Recursion and Induction

Exercise 1(a)

Let $M = (\top, \top, \ldots, \top)$. We'll show by induction that for any sentence, if A has no occurrences of $\bot$, then $[\![A]\!]_M = \top$. There are four cases to consider.

1. For any sentence letter s, $[\![s]\!]_M = M(s) = \top$.
2. For $\bot$, the conditional is trivially true.
3. If $B \wedge C$ contains no occurrences of $\bot$, then B and C each contain no occurrences of $\bot$. So, by the inductive hypothesis, $[\![B]\!]_M = [\![C]\!]_M = \top$. It follows that

$$[\![B \wedge C]\!]_M = [\![B]\!]_M \wedge [\![C]\!]_M = \top \wedge \top = \top \tag{1}$$

4. For the case $B \to C$, as in case 3 we can assume that $[\![B]\!]_M = [\![C]\!]_M = \top$. Then

$$[\![B \to C]\!]_M = [\![B]\!]_M \to [\![C]\!]_M = \top \to \top = \top \tag{2}$$

So we conclude by induction that for every sentence A, $[\![A]\!]_M = \top$. This means that $[\![A]\!]$ is positive.

Exercise 1(b)

Consider the truth-function $\bot$ that takes every $M \in 2^S$ to $\bot$. By part (a), for any sentence A that contains no $\bot$, then $[\![A]\!]_M \neq \bot(M)$ when $M = (\top, \ldots, \top)$, and so $[\![A]\!] \neq \bot$.

Exercise 2

Let B be any sentence, and let M be any model. We'll show by induction that for any sentence $A \in \mathsf{Prop}\,1$, if A contains no conditionals, if $[\![A[B]]\!]_M = \top$, then $[\![B]\!] = \top$. There are four cases to consider.

1. For the sentence letter p, $p[B] = B$. So clearly if $[\![p[B]]\!]_M = \top$ then $[\![B]\!]_M = \top$.
2. $[\![\bot[B]]\!]_M = [\![\bot]\!]_M = \bot$. This can't be $\top$, so our conclusion holds trivially in this case.

3. If a conjunction $A_1 \wedge A_2$ contains no conditionals, then neither does A_1 or A_2. So the inductive hypothesis says:

$$\begin{aligned}&\text{If } [\![A_1[B]]\!]_M = \top \text{ then } [\![B]\!]_M = \top\\ &\text{If } [\![A_2[B]]\!]_M = \top \text{ then } [\![B]\!]_M = \top\end{aligned} \tag{3}$$

   Using the definitions of substitution and the interpretation function:

$$[\![(A_1 \wedge A_2)[B]]\!]_M = [\![A_1[B] \wedge A_2[B]]\!]_M = [\![A_1[B]]\!]_M \wedge [\![A_2[B]]\!]_M \tag{4}$$

   So if $[\![(A_1 \wedge A_2)[B]]\!]_M = \top$ then $[\![A_1[B]]\!]_M = \top$ and $[\![A_2[B]]\!]_M = \top$. Therefore, $[\![B]\!]_M = \top$, by the inductive hypothesis, which is what we needed to show.

4. The conditional $A_1 \to A_2$ obviously contains a conditional, so again our conclusion holds trivially.

It follows that for every model, if $[\![A[B]]\!]_M = \top$ then $[\![B]\!]_M = \top$. This means that $A[B] \vDash B$.

Exercise 3(a)

1. $c(s) = 0$ for any $s \in S$
2. $c(\bot) = 0$
3. $c(A \wedge B) = c(A) + c(B) + 1$
4. $c(A \to B) = c(A) + c(B)$

Exercise 3(b)

For the inductive proof there are four cases.

1. For a sentence letter $s \in S$, $c(s) = 0$, and since $c(s[B])$ is a natural number, $c(s[B]) \geq c(s)$.
2. $c(\bot) = 0$, so the same reasoning applies as case 1.

3. By the definitions of substitution and c,

$$c((A_1 \to A_2)[B]) = c(A_1[B] \to A_2[B]) = c(A_1[B]) + c(A_2[B]) \tag{5}$$

   By the inductive hypothesis, $c(A_1[B]) \geq c(A_1)$ and $c(A_2[B]) \geq c(A_2)$. It follows that

$$c(A_1[B]) + c(A_2[B]) \geq c(A_1) + c(A_2) = c(A_1 \to A_2) \tag{6}$$

   Putting these together, we see that

$$c((A_1 \to A_2)[B]) \geq c(A_1 \to A_2) \tag{7}$$

4. By the definitions of substitution and c,

$$c((A_1 \wedge A_2)[B]) = c(A_1[B] \wedge A_2[B]) = c(A_1[B]) + c(A_2[B]) + 1 \tag{8}$$

   By the inductive hypothesis, $c(A_1[B]) \geq c(A_1)$ and $c(A_2[B]) \geq c(A_2)$. It follows that

$$c(A_1[B]) + c(A_2[B]) + 1 \geq c(A_1) + c(A_2) + 1 = c(A_1 \wedge A_2) \tag{9}$$

   Putting these together,

$$c((A_1 \wedge A_2)[B]) \geq c(A_1 \wedge A_2) \tag{10}$$

Exercise 3(c)

There are four cases.

1. For a sentence letter $t \in S$,

$$\#(s, t) = \begin{cases} 1 & \text{if } s = t \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

2. $\#(s, \bot) = 0$
3. $\#(s, A \wedge B) = \#(s, A) + \#(s, B)$

4. $\#(s, A \to B) = \#(s, A) + \#(s, B)$

Exercise 3(d)

There are four cases to consider for an inductive proof.

1. By the definition of substitution, $p[B] = B$. Also $c(p) = 0$, and $\#(p, p) = 1$. So:

$$c(p[B]) = c(B) = 0 + c(B) \cdot 1 = c(p) + c(B) \cdot \#(p, p) \tag{12}$$

2. By the definitions, $\bot[B] = \bot$, $c(\bot) = 0$, and $\#(p, \bot) = 0$. So:

$$c(\bot[B]) = c(\bot) = 0 = 0 + c(B) \cdot 0 = c(\bot) + c(B) \cdot \#(p, \bot) \tag{13}$$

3. For a conjunction $A_1 \wedge A_2$, the inductive hypothesis tells us:

$$\begin{aligned} c(A_1[B]) &= c(A_1) + c(B) \cdot \#(p, A_1) \\ c(A_2[B]) &= c(A_2) + c(B) \cdot \#(p, A) \end{aligned} \tag{14}$$

By the definitions,

$$c((A_1 \wedge A_2)[B]) = c(A_1[B] \wedge A_2[B]) = c(A_1[B]) + c(A_2[B]) + 1 \tag{15}$$

Applying the inductive hypothesis, this is equal to

$$c(A_1) + c(B) \cdot \#(p, A_1) + c(A_2) + c(B) \cdot \#(p, A_2) + 1 \tag{16}$$

Rearranging terms, this is equal to

$$(c(A_1) + c(A_2) + 1) + c(B) \cdot (\#(p, A_1) + \#(p, A_2)) \tag{17}$$

And by the definitions of c and # this is the same as

$$c(A_1 \wedge A_2) + c(B) \cdot \#(p, A_1 \wedge A_2) \tag{18}$$

4. Similar to case 3.

Exercise 4(a)

$$\begin{aligned} \mathrm{sl}(s) &= \{s\} \quad \text{for a sentence letter } s \in S \\ \mathrm{sl}(\bot) &= \{\} \\ \mathrm{sl}(A \wedge B) &= \mathrm{sl}\,A \cup \mathrm{sl}\,B \\ \mathrm{sl}(A \to B) &= \mathrm{sl}\,A \cup \mathrm{sl}\,B \end{aligned} \tag{19}$$

Exercise 4(b)

We'll show by induction that for every sentence A, if M and M' have the same truth value for every sentence letter in A, then $[\![A]\!]_M = [\![A]\!]_{M'}$. There are four cases to consider!

1. For a sentence letter s, since obviously $s \in \mathrm{sl}\,s = \{s\}$,

$$[\![s]\!]_M = M(s) = M'(s) = [\![s]\!]_{M'} \tag{20}$$

2. $[\![\bot]\!]_M = \bot = [\![\bot]\!]_{M'}$

3. Consider a conjunction $A \wedge B$. If $Ms = M's$ for every \$s □ $\mathrm{sl}(A \wedge B) = \mathrm{sl}\,A \cup \mathrm{sl}\,B$, it follows that $Ms = M's$ for every $s \in \mathrm{sl}\,A$, and also $Ms = M's$ for every $s \in \mathrm{sl}\,B$. So by the inductive hypothesis,

$$\begin{aligned} [\![A]\!]_M &= [\![A]\!]_{M'} \\ [\![B]\!]_M &= [\![B]\!]_{M'} \end{aligned} \tag{21}$$

   Then we can conclude:

$$[\![A \wedge B]\!]_M = [\![A]\!]_M \wedge [\![B]\!]_M = [\![A]\!]_{M'} \wedge [\![B]\!]_{M'} = [\![A \wedge B]\!]_{M'} \tag{22}$$

4. Similar to case 3.

Exercise 4(c)

If neither $A \vDash \bot$ nor $\vDash B$, then there must be some model M such that $[\![A]\!]_M = \top$, and there must be some model M' such that $[\![B]\!]_{M'}$. In that case, define a new model M^*.

$$M^*(s) = \begin{cases} M(s) & \text{if } s \neq p \\ M'(s) & \text{otherwise} \end{cases} \tag{23}$$

Note that for every $s \neq p$, $M^*(s) = M(s)$. Since $p \notin \mathrm{sl}\,A$, it follows that $M(s) = M^*(s)$ for every $s \in \mathrm{sl}\,A$. So by part (b),

$$[\![A]\!]_{M^*} = [\![A]\!]_M = \top \tag{24}$$

Similarly, since $\mathrm{sl}\,A = \{p\}$, and $M'(p) = M^*(p)$, by part (b) again,

$$[\![B]\!]_{M^*} = [\![B]\!]_{M'} = \bot \tag{25}$$

So M^* is a model in which A is true and B is false. So it is not the case that $A \vDash B$. Contrapositively, if $A \vDash B$, then either $A \vDash \bot$ or else $\vDash B$.

Note the proof works with hardly any changes if instead we just assume:

$$\mathrm{sl}\,A \cap \mathrm{sl}\,B = \{\} \tag{26}$$