An Adaptive Algorithm for Selecting Profitable Keywords for Search-Based Advertising Services

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ABSTRACT

Increases in online search activities have spurred the growth of search-based advertising services offered by search engines. These services enable companies to promote their products to consumers based on their search queries. In most search-based advertising services, a company selects a set of keywords, determines a bid price for each keyword, and designates an ad associated with each selected keyword. When a consumer searches for one of the selected keywords, search engines then display the ads associated with the highest bids for that keyword on the search result page. A company whose ad is displayed pays the search engine only when the consumer clicks on the ad. With millions of available keywords and a highly uncertain clickthru rate associated with the ad for each keyword, identifying the most effective set of keywords and determining corresponding bid prices becomes challenging for companies wishing to promote their goods and services via search-based advertising.

Motivated by these challenges, we formulate a model of keyword selection in search-based advertising services. We develop an algorithm that adaptively identifies the set of keywords to bid on based on historical performance. The algorithm prioritizes keywords based on a prefix ordering—sorting of keywords in a descending order of profit-to-cost ratio. We show that the average expected profit generated by the algorithm converges to near-optimal profits. Furthermore, the convergence rate is independent of the number of keywords and scales gracefully with the problem’s parameters. Extensive numerical simulations show that our algorithm outperforms existing methods, increasing profits by about 7%.

1. INTRODUCTION

Search-based advertising services offered by search engines enable companies to promote their products to targeted groups of consumers based on their search queries. Examples of search-based advertising services include Google Adwords (http://www.google.com/adwords), Yahoo! Sponsored Search (http://searchmarketing.yahoo.com/srch/index.php), and Microsoft’s forthcoming MSN AdCenter (http://advertising.msn.com/searchadv/). In most search-based advertising services, a company wishing to advertise sets a daily budget, selects a set of keywords, and determines the bid price for each selected keyword. When consumers search for one of the selected keywords, search engines then display the ads associated with the highest bids for that keyword on the search result page. A company whose ad is displayed pays the search engine only when the consumer clicks on the ad. Note that the ad is not displayed if the company’s spending that day has exceeded its budget. Figure 3 in Appendix A shows examples of the ads shown to a user under the search-based advertising programs offered by Google and Yahoo!

Details of the actual cost of each keyword, of how a search query matches to a keyword, and of how different bid amounts translate to the position on the search result page vary from one search engine to another. Under the Google Adwords program, for instance, each click will cost the ad’s owner an amount equals to the next highest bid plus one cent. Moreover, when determining the ad’s placement on the search result page, the Google Adwords program takes into account not only the bid amount, but also the ad’s popularity and clickthrus. On the other hand, under Yahoo! Sponsored Search program, the highest bidder receives the top spot.

With millions of available keywords, a user of search-based advertising services faces challenging decisions. The user not only has to identify an effective set of keywords, she also needs to develop ads that will draw consumers and to determine a bid price for each keyword that balances the tradeoffs between the costs and profits that may result from each click on the ad associated with the keyword. Furthermore, much of the information that is required to compute these tradeoffs is not known in advance and is difficult to estimate a priori. For instance, the clickthru probability associated with the ad for each keyword—the probability that the consumer will click on the ad once it appears on the search result page—varies dramatically depending on the keyword, the ad itself, and the position of the ad on the search result page.

In this paper, we consider one of the challenges faced by users of search-based advertising services: given a fixed daily budget and unknown clickthru probabilities, develop a policy that adaptively selects a set of keywords in order to maximize total expected profits. We assume that we have \(N\) keywords indexed by \(1, 2, \ldots, N\). For any \(A \subseteq \{1, \ldots, N\}\), let the random variable \(Z_A\) denote the profits from selecting
Identifying profitable keywords requires knowledge of the clickthru probability of the ad associated with each keyword. Unfortunately, the clickthru rate is generally not known in advance and we can obtain an estimate of its value only by selecting the keyword and observing the resulting displays of the ad – commonly referred to as its impressions – and the number of clicks on it. Since we pay for each click on the ad, this process can potentially result in significant costs, yet may offer an opportunity to discover potentially profitable keywords. We thus must balance the tradeoffs between selecting keywords that seems to yield high average profits based on past performance and selecting previously unused keywords in order to learn about their clickthru probabilities. This is usually cast in the machine learning literature as balancing “exploitation” (of known good options) and “exploration” (of unknown options that might be better than known options).

We model the problem as follows: we assume that in every time period, some number of queries arrive, where the number is independent and identically distributed. At the beginning of each time period, we must specify a subset of keywords on which we will bid. Then queries arrive sequentially; keyword $i$ is queried with probability $\lambda_i$. If we have enough money remaining in our budget, our ad is displayed. With some probability $p_i$, the user clicks on our ad; we receive profit $\pi_i$ and must pay cost $c_i$. We assume that the probabilities $\lambda_i$, costs $c_i$, and profits $\pi_i$ are known; justification of our ability to know the first two quantities from publicly available sources is given in Section 2.1. The probabilities $p_i$ are unknown, and we learn them over time. Our goal is to give an algorithm that converges to near-optimal profits as the number of time periods tends towards infinity.

We begin by considering the static case – when the probabilities $p_i$ are known. This problem is related to the well-known stochastic knapsack problem and is NP-complete. Following the approximation algorithms of Sahni [23] for the standard knapsack problem and the nonadaptive algorithm of Dean, Goemans, and Vondrak [7] for the stochastic knapsack problem, we show that we can obtain a near-optimal approximation algorithm by considering prefix-orderings. We order the keywords in decreasing order of profit-to-cost ratio; that is, $\pi_1/c_1 \geq \pi_2/c_2 \geq \cdots \geq \pi_N/c_N$. Then our algorithm chooses a set of the form $\{1, \ldots, \ell\}$ for some integer $\ell$; we call such a set a prefix of the keywords. We show that if the cost of each item is sufficiently small compared to our budget, if the expected number of arrivals of any given keyword is not too large, and if the expected number of searched keywords is close to its mean, then in expectation our algorithm returns a near-optimal solution, where the closeness to optimality depends on how well these assumptions are satisfied. We give evidence that in practice these assumptions are quite likely to be true.

Note that the static case of our problem is somewhat different than the stochastic knapsack problem. In that problem, profits are known, but sizes are drawn from arbitrary known distributions and items can be placed in the knapsack in a specified order. Here costs (corresponding to item sizes) are deterministic and known, but query arrival (corresponding to item placement) is random and not under our control.

Our result in the static case guides the development of an adaptive algorithm for the case when the clickthru probabilities $p_i$ are unknown and we must learn them over time. In each time period, we either choose a random prefix of keywords, or a prefix of keywords using our algorithm from the static case applied to our estimates of the $p_i$ from the last time period. We show that, averaged over the number of time periods $T$, in expectation our algorithm converges to near-optimal profits, where the performance guarantee is almost the same as that of the static case modulo some terms that vanish as $T$ goes to infinity.

This result should be compared to traditional multi-armed bandit algorithms [3, 4, 8, 15, 16, 17]. These algorithms must choose one of $N$ slot machines (or “bandits”) to play in each time step. Each play of a bandit will yield a reward, whose distribution is not known in advance. Several algorithms have been proposed for the multi-armed bandit problem whose average expected reward also converges to the optimal. If we associate each bandit with a prefix of the keywords, we can show that in expectation these algorithms converge to near-optimal profits with the same performance guarantee as our static case. The convergence rates of these algorithms, however, depend on the number of possible keywords $N$, which might be extremely large in our case. In contrast, the convergence rate of our prefix-based adaptive algorithm is independent of $N$; it instead depends primarily on the value of the largest $\ell$ such that the expected cost of the prefix $\{1, \ldots, \ell\}$ does not exceed our budget. We believe that in practice the value of $\ell$ will be significantly smaller than the total number of keywords $N$. Furthermore, our dependence on $T$ is much better than the multi-armed bandit algorithms, so that our algorithm converges faster. Finally, when compared to two standard multi-armed bandit algorithms, extensive numerical simulations show that in as little as 30 time periods our algorithm outperforms these two algorithms, yielding significantly better profits. In one set of simulations, our profits were about 7% better, and in another about 20% better.

To the best of our knowledge, this work is the first publicly available study that addresses the problem of identifying profitable sets of keywords, realistically taking into account the uncertainty of the clickthru probability and the need to estimate its value based on historical performance. While there has been a great deal of interest from researchers in the area of search-based advertising services, much of it has focused on the design of auctions and payment mechanisms [1, 5, 6, 9, 11, 18, 21]. Our paper is one of only a few that focuses on the users of search-based advertising. Kitts et al. [12] consider the problem of finding optimal bidding strategies under a variety of objectives, but only in our static setting in which the clickthru probabilities are known in advance. We believe that removing this assumption is a crucial step towards a usable algorithm. Furthermore, Kitts et al. do not offer an algorithm for finding an optimal set of keywords when the objective is to maximize expected profit subject to a budget constraint, which again is needed given the requirements of using a service such as Google Adwords.

We have attempted to construct a model of the problem that is simultaneously tractable and realistic, justifying our
choices with real data whenever possible. Our resulting algorithm is simple to state and code, and provides good results in simulations, beating out several potential competitors. Our paper is structured as follows. We present our model in Section 2. We analyze the static case in Section 3. We discuss the connection of our problem with multi-armed bandits in Section 4.1, then give our algorithm in Section 4.2. We give the results of an experimental study comparing our algorithm with multi-armed bandit algorithms in Section 5. We discuss some possible extensions of our model in Section 6. Because of space constraints, many proofs and supporting figures have been placed in the appendix.

We conclude our introduction by mentioning some other related literature. During the past few years, there has been resurgent interest in the study of online decision making with extremely large decision spaces [2, 13, 14, 24]. However, the algorithms developed in these papers are problem-specific and they are not applicable to the problem considered here.

We should note that Mehta et al. [19] have considered the problem faced by a search engine on how to optimally match incoming queries with the bid placed by advertisers. This problem is very different from the one studied in this paper; our focus is on the company trying to advertise rather than the search engine providing the advertising service.

2. PROBLEM FORMULATION AND MODEL DESCRIPTION

In this section, we develop a model of search-based advertising services and formulate the problem of identifying profitable sets of keywords. Sections 2.1 and 6 provide additional discussion of the problem formulation, justification of the model’s assumptions, and discussion of the publicly available data sources that can be used to estimate some of the model’s parameters. Assume that we have a probability space $\Omega, \mathcal{F}, \mathbb{P}$ and $N$ keywords indexed by $1, 2, \ldots, N$. For any $t \geq 1$, let $S^t$ denote the total number of search queries that arrive in period $t$. We assume that $S^1, S^2, \ldots$ are independent and identically distributed random variables with mean $1 < \mu < \infty$ and the distribution of $S^t$ is known in advance.

We assume that, in each time period, the search queries arrive sequentially, and each query can correspond to any one of the $N$ keywords. For any $t \geq 1$, let $Q^t_i$ denote the random variable associated with the $i$th search query in period $t$. We assume that $Q^t_i$ is independent and identically distributed random variables and $P\{Q^t_i = i\} = \lambda_i$ for $i = 1, \ldots, N$, where $\sum_{i=1}^{N} \lambda_i = 1$. We assume that $\lambda_i$’s are known in advance. In Section 2.1, we will show an approach for estimating the distribution of $S^t$ and the parameters $\lambda_i$ using publicly available information provided by search engines.

For each keyword $1 \leq i \leq N$, let $0 \leq \pi_i \leq 1$, $c_i > 0$, and $\pi_i \geq 0$ denote the clickthru probability, the ad cost or cost-per-click (CPC), and the expected profit (per click) of keyword $i$, respectively. We will assume that the CPC and the expected profits ($c_i$ and $\pi_i$) are known in advance and remain constant. Although the CPC of each keyword depends on the bidding behaviors of other users, we believe this requirement is a reasonable model in the short-term. In Section 2.1, we will provide additional discussion of this assumption and show how we can approximate the current CPC associated with each keyword using publicly available information provided by search engines.

At the beginning of each time period, we start with a balance of $B$ dollars and must select a set of keywords, which will determine the set of ads that will be shown to consumers. As a search query arrives, if the query matches with one of the selected keywords and we have enough money remaining in the account, the ad associated with the keyword appears on the search result page. Each display of an ad is called an impression. If the consumer then clicks on the ad, we receive a profit, pay the cost associated with the keyword to the search engine, update our remaining balance, and wait for an arrival of another search query. The process terminates once all search queries have arrived for the time period.

By appropriate scaling, we may assume without loss of generality that the beginning balance in each time period is $\$1. Let $\{1, \ldots, N\}$ denote the set of possible keywords on which we can bid. Let $1(\cdot)$ denote the indicator function and let the random variable $B^t_i$ indicate the remaining balance when the $r^{th}$ search query appears and we have decided to select keywords in the set $A_t \subseteq \{1, \ldots, N\}$ in period $t$. Let $X_{r,t}$ be a Bernoulli random variable with parameter $p_{r,t}$ indicating whether or not the consumer clicks on the ad associated with keyword $i$ during the arrival of the $r^{th}$ search query in period $t$. If we select the set of keywords $A_t \subseteq \{1, \ldots, N\}$ in period $t$, the expected profit $E[Z_{A_t}]$ is given by

$$E[Z_{A_t}] = E \left[ \sum_{i=1}^{N} \pi_i 1 \left( i \in A_t, Q^t_i = i, B^t_{A_t} \geq c_i, X_{r,t} = 1 \right) \right].$$

We assume that, for any $i$, the random variables $(X_{r,t}^i : t \geq 1, r \geq 1)$ are independent and identically distributed and are independent of the arrival process.

The above expression for the expected profit reflects our model of the search-based advertising dynamics. Under this model, we receive a profit from keyword $i$ during the arrival of the $r^{th}$ search query in period $t$ if and only if

(a) We bid on keyword $i$ at the beginning of period $t$, i.e. $i \in A_t$;

(b) The $r^{th}$ query corresponds to keyword $i$, i.e. $Q^t_i = i$;

(c) We have enough balance in the account to pay for the cost of the keyword if the consumer clicks on the ad, i.e. $B^t_{A_t} \geq c_i$;

(d) The consumer actually clicks on the ad, i.e. $X_{r,t}^i = 1$.

As mentioned in the previous section, the clickthru rate $p_{i}$ for any keyword $i$ is generally not known in advance and we can obtain an estimate of its value only by selecting the keywords and observing the resulting impressions and clicks. Therefore, we must balance the tradeoffs between choosing keywords that seem to yield high average profits based on past performance and trying previously unused keywords in order to learn about their clickthru probabilities.

To analyze the tradeoffs between exploitations and explorations, let us introduce the following notation. Let $(F_t : t \geq 1)$ denote a filtration on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $F_t$ corresponds to all events that have happened up to time $t$. A policy $\phi = (A_1, A_2, \ldots)$ is a sequence of random variables, where $A_t \subseteq \{1, 2, \ldots, N\}$ corresponds to the set of keywords selected in period $t$. We will focus exclusively on a class of non-anticipatory policies defined as follows.
DEFINITION 1. A policy \( \phi = (A_1, A_2, \ldots) \) is non-anticipatory if \( A_t \) is \( F_t \)-measurable for all \( t \geq 1 \). Let \( \Pi \) denote the collection of all non-anticipatory policies.

For any \( T \geq 1 \), we aim to find a non-anticipatory policy that maximizes the expected profit over \( T \) periods, i.e.,

\[
\sup_{\phi \in \Pi} \mathbb{E} \left[ \sum_{t=1}^{T} E[Z_{A_t}] \right].
\]

In general, the above optimization is intractable and we aim to find an approximation algorithm that yields near optimal expected profit.

2.1 Model Calibration

Although we aim to estimate adaptively the clickthru probability associated with the ad of each keyword based on past performance, our formulation assumes that we can estimate \textit{a priori} the arrival process, the query distributions, and the current CPC of each keyword. In this section, we identify publicly available data sources that might be used to estimate these parameters.

Figure 4(a) in Appendix A shows the estimated number of search queries in Yahoo! in September 2005 related to the keyword “cheap Europe trip.” This data is obtained through the publicly available Keyword Selector tool provided by the Yahoo! Sponsored Search program (http://inventory.overture.com/d/searchinventory/suggestion/). Using this information, it might be possible to estimate the probability that each query will correspond to a particular keyword.

Our model also assumes that the cost-per-click (CPC) associated with each keyword is known in advance. Although the actual CPC depends on the outcome of the bidding behaviors of other users, as indicated in Section 2, we believe that this requirement is a reasonable assumption in the short-term. Whereas the clickthru probability associated with the ad of each keyword can be estimated only by bidding on the keywords and observing the resulting impressions and clicks, it is possible to estimate the CPC associated with each keyword using data provided by search engines. For instance, Figure 4(b) shows an example of the current outstanding bids for a keyword, “cheap Europe trip” under the Yahoo! Sponsored Search Market-  

performance, our formulation assumes that we can estimate the probability associated with the ad of each keyword based on past information. For any \( T \geq 1, \) we aim to find a non-anticipatory policy in the next section. In general, the above optimization is NP-complete [7, 10]. Unless \( P = NP \), it is thus unlikely that there exists a polynomial-time algorithm for solving the Static Bidding problem. In this section, we identify a special structure of our problem that will enable the development of an efficient approximation algorithm in Section 3.2. Throughout this paper, we will make the following assumptions on the ordering of keywords and their costs.

Assumption 1.

(a) The keywords are indexed such that \( \pi_1/c_1 \geq \pi_2/c_2 \geq \cdots \geq \pi_N/c_N \).

(b) \( \sum_{i=1}^{N} c_i \lambda_i > 1 \).

(c) There exists \( k \geq 1 \) and \( 0 \leq \alpha < 1 \) such that \( c_i \leq 1/k \) and \( \lambda_i \mu \leq k^\alpha \) for all \( i \).

Before we proceed to the statement of our theorem, let us first discuss each of these assumptions. The first two assumptions are without loss of generality. Assumption 1(a) assumes that the keywords are sorted by a prefix ordering – in a descending order of profit-to-cost ratio. Moreover, by adding fictitious keywords with zero profit, we can assume without loss of generality that \( \sum_{i=1}^{N} c_i \lambda_i > 1 \), which is the statement of Assumption 1(b).

Thus our key assumption is Assumption 1(c). This assumption places constraints on the magnitude of the cost and the expected number of arrivals associated with each keyword. The requirement that \( c_i \leq 1/k \) implies that each click of the ad associated with keyword \( i \) will cost at most \( 1/k \) fraction of the total budget. As demonstrated in Figure 5 in Appendix A, we believe that this assumption is a reasonable requirement. Figure 5 shows the distribution of the median bid price for each of the 842 travel-related keywords. These keywords were chosen based on the authors’ experience working with an online seller of travel packages. Examples of the keywords include “hawaii vacation package”, “africa safari”, and “london discount airfare”. A complete list of 842 keywords used in this study is available at [http://uv.bidtool.overture.com/d/search/tools/bidtool/index.jhtml](http://uv.bidtool.overture.com/d/search/tools/bidtool/index.jhtml). As seen from the figure, the average median bid follows directly from the general expression given in Equation 1 in Section 2; we simply drop the superscript \( t \) denoting the time period. The following lemma allows us to simplify the objective function of the Static Bidding problem; its proof appears in Appendix B.

LEMMA 1. For any \( A \subseteq \{1, 2, \ldots, N\}, r \geq 1, \) and keyword \( i \in A \),

\[
P\{Q_r = i, B_r^A \geq c_i, X_{ri+1} = 1 | S \geq r \} = \lambda_p \mu \mathbb{P} \{ B_r^A \geq c_i, S \geq r \}
\]

and

\[
E[Z_A] = \sum_{i \in A} \pi_i \lambda_i \mathbb{P} \left( \sum_{r=1}^{\infty} E[1 \{ B_r^A \geq c_i, S \geq r \}] \right).
\]
Theorem 1. Let $k$ and $\alpha$ be defined as in Assumption 1. Suppose the probabilities $p_i$ are known and $1/k^{1-\alpha} \leq 1 - 1/k - 1/k^{(1-\alpha)/3}$. Let $I_\gamma$ be defined by

$$I_\gamma = \max \left\{ \ell : \mu \sum_{i=1}^\ell c_i p_i \lambda_i \leq 1 - \frac{1}{k} - \frac{1}{k^{(1-\alpha)/3}} \right\}.$$

If $\rho = E[\min \{S, \mu\}] / \mu$, measures how tightly the distribution of $S$ concentrates around its mean. In the most important example where $S$ follows a Poisson distribution, this ratio can be computed explicitly and it is given in the following lemma, whose proof appears in Appendix C.

Lemma 2. If $S$ is a Poisson random variable with mean $\mu > 1$, then $1 - 3/\sqrt{\pi} \leq E[\min \{S, \mu\}] / \mu \leq 1$.

In most applications, the expected number of search queries $\mu$ will be very large, ranging from hundreds of thousands to over millions of queries per day. If the arrival follows a Poisson distribution, then we see that the ratio $E[\min \{S, \mu\}] / \mu$ will be very close to one.

The second factor that influences the quality of our approximation algorithm is the value of $k$ and $\alpha$. Since $0 \leq \alpha < 1$, as $k$ – the relative value between the budget and the CPC for each keyword – increases, the expected profit will be close to the optimal. As indicated in the previous section, we believe that this is a reasonable assumption. If the budget for a campaign ranges from $200-$400 per day and the maximum CPC for each keyword ranges from 30 to 50 cents, this translates to a value of $k$ from 400 to 1200.

In addition, for our approximation algorithm to perform well, the value $\alpha$ should be small. Recall from Assumption 1 that for any keyword $i$, $\lambda_i \mu \leq k^\alpha$. Thus, $\alpha$ measures the average number of search queries for each keyword $i$. As shown in Figure 6 in Appendix A, most keywords have an average of less than 20 queries per day, leading us to believe that this requirement should hold for a large proportion of keywords.

Figure 1 shows the value of the performance bound given in Theorem 1 (with $\rho = 1.0$) for various values of $k$ and $\alpha$, with $k$ ranging from $10^3$ to $10^6$. We observe, for instance, that if $k = 1000$, $\alpha = 0.2$, and $\rho = 1$, Theorem 1 shows that the algorithm is within 0.68 of the optimal profit.
\[ \sum_{i=1}^{N} c_i \lambda_i z_i \leq 1 \text{ and also that} \]
\[ E[Z_A] = E \left[ \sum_{r=1}^{S} \prod_{i=1}^{N} \mathbb{1} \left( i \in A, Q_r = i, B_r^A \geq c_i, X_{r1} = 1 \right) \right] = \sum_{i \in A} \pi_i \lambda_i \left( \sum_{r=1}^{\infty} E \left[ 1 \left( B_r^A \geq c_i, S \geq r \right) \right] \right) = \sum_{i=1}^{N} \pi_i \lambda_i \rho(z_i) \]

Since \( A \) is arbitrary, the desired result follows. \qed

The next lemma relates the optimal value \( Z_{LP} \) of the linear program defined in Lemma 3 to the expected profits obtained when we select keywords according to the prefix ordering; its proof is in Appendix D.

**Lemma 4.** Let \( H = \{1, \ldots, u\} \) where \( u \) is any positive integer such that \( 0 < u \sum_{i=1}^{N} c_i \lambda_i \rho(z_i) \leq 1 - 1/k - 1/k^{(1-\alpha)/3} \), so that \( u \leq I_U \). Let \( Z_{LP} \) denote the optimal value of the linear program defined in Lemma 3. Then,
\[ \sum_{i \in H} \pi_i \lambda_i \rho(z_i) \geq \left( \sum_{i \in H} c_i \lambda_i \rho(z_i) \right) Z_{LP} \geq \left( \sum_{i \in H} c_i \lambda_i \rho(z_i) \right) Z^* . \]

The final lemma provides a lower bound on the probability \( \mathcal{P} \{ B_r^U \geq c_i | S \geq r \} \) that we have enough money remaining if the \( k \)-th search query corresponds to keyword \( i \), given that we bid on a set of keywords \( H \) and \( r \leq \mu \). Since the proof is fairly technical, the details are given in Appendix E.

**Lemma 5.** Let \( H = \{1, \ldots, u\} \) where \( u \) is any positive integer such that \( 0 < u \sum_{i=1}^{N} c_i \lambda_i \rho(z_i) \leq 1 - 1/k - 1/k^{(1-\alpha)/3} \), so that \( u \leq I_U \). Then, for any keyword \( i \in H \) and \( 1 \leq r \leq \mu \),
\[ \mathcal{P} \{ B_r^U \geq c_i | S \geq r \} \geq 1 - \frac{k}{k^{(1-2\alpha)/3} - \mu} \sum_{i \in H} c_i \lambda_i \rho(z_i) \]
\[ \geq 1 - \frac{1}{k^{(1-2\alpha)/3}} . \]

Here is the proof of Theorem 1.

**Proof.** Note that since by hypothesis \( 1/k^{(1-\alpha)} \leq 1 - 1/k - 1/k^{(1-\alpha)/3} \), it follows that \( I_U \) is well-defined. It follows from Lemma 5 that for any keyword \( i \in P \) and \( 1 \leq r \leq \mu \),
\[ \mathcal{P} \{ B_r^P \geq c_i | S \geq r \} \geq 1 - 1/k^{(1+2\alpha)/3} . \]

Therefore,
\[ E[Z_P] = \sum_{i \in P} \pi_i \lambda_i \rho(z_i) \prod_{r=1}^{\infty} \mathcal{P} \{ B_r^P \geq c_i, S \geq r \} \]
\[ \geq \sum_{i \in P} \pi_i \lambda_i \rho(z_i) \prod_{r=1}^{\infty} \mathcal{P} \{ B_r^P \geq c_i, S \geq r \} \]
\[ \geq \left( 1 - \frac{1}{k^{(1+2\alpha)/3}} \right) \sum_{i \in P} \mathcal{P} \{ S \geq r \} \prod_{i \in P} \pi_i \lambda_i \rho(z_i) \]
\[ = E[\min \{ S, \mu \}] \mu \left( 1 - \frac{1}{k^{(1+2\alpha)/3}} \right) \sum_{i \in P} \pi_i \lambda_i \rho(z_i) \]
\[ \geq E[\min \{ S, \mu \}] \mu \left( 1 - \frac{1}{k^{(1+2\alpha)/3}} \right) \sum_{i \in P} \pi_i \lambda_i \rho(z_i) \]
\[ \geq E[\min \{ S, \mu \}] \mu \left( 1 - \frac{1}{k^{(1+2\alpha)/3}} \right) - \frac{2}{k^{(1-\alpha)/3}} \right) \]

where the second inequality follows from Lemma 5. The third inequality follows from Lemma 2. The final inequality follows from Assumption 11(c), which implies that, for any keyword \( i \), \( \mu \sum_{i=1}^{N} c_i \lambda_i \rho(z_i) \geq 1/k - 1/k^{(1-\alpha)/3} - 1/k^{(1-\alpha)/3} \).

Since \( k \geq 1 \) and \( 0 < \alpha < 1 \), \( 1/k^{(1-\alpha)/3} \geq 1/k - 1/k^{(1-\alpha)/3} \), we have that
\[ \left( 1 - \frac{1}{k^{(1+2\alpha)/3}} \right) \left( 1 - \frac{1}{k - \frac{2}{k^{(1-\alpha)/3}}} \right) \geq 1 - \frac{1}{k^{(1+2\alpha)/3}} - \frac{2}{k^{(1-\alpha)/3}} - \frac{1}{k} \]
\[ \geq 1 - \frac{1}{k^{(1+2\alpha)/3}} - \frac{2}{k^{(1-\alpha)/3}} - \frac{1}{k} \]

which is the desired result. \qed

### 4. AN ADAPTIVE SETTING: WHEN THE \( P_i \) ARE UNKNOWN

In this section, we consider the case when the click-through probabilities are not known in advance. To estimate these probabilities, we need to select the keywords and observe the resulting impressions and clicks, a process that can potentially result in significant costs. We thus need to carefully balance the tradeoffs between explorations and exploitations. In Section 4.1, we will show how this problem can be formulated as a multi-armed bandit problem, where each bandit corresponds to a subset of keywords. We then aim to develop a strategy that adaptively selects bandits that yield near-optimal profits.

By formulating the problem an instance of the multi-armed bandit problem, we can apply existing adaptive algorithms developed for this class of problems. Unfortunately, existing multi-armed bandit algorithms have performance guarantees that deteriorate as the number of keywords increases, motivating us to develop an alternative adaptive approximation algorithm that leverages the special structure of our problem.

In Section 4.2, we describe our improved algorithm and show that the algorithm exhibits a superior performance bound compared with traditional multi-armed bandit algorithms. We will show that the quality of the solution generated by our algorithm is independent of the number of keywords and scales gracefully with the problem’s other parameters. Then, in Section 5, we show through extensive simulations that our algorithm outperforms traditional multi-armed bandit algorithms, increasing profits by about 7%.

#### 4.1 A Traditional Multi-Armed Bandit Approach

Theorem 1 shows that by considering only subsets of keywords that follow the prefix ordering – a descending order of profit-to-cost ratio – we can achieve near-optimal profits. This result enables us to reduce the size of the decision space from \( 2^N \) (all possible subsets of \( N \) keywords) to just \( N \).

Assume that the keywords are indexed as in Assumption 1. For any \( 1 \leq i \leq N \), let \( D_i = \{1, 2, \ldots, i\} \). We can view each \( D_i \) as a decision whose payoff corresponds to the expected profit that results from selecting keywords in \( D_i \), i.e., \( E[Z_{D_i}] \). Let \( Z^\text{pref} \) denote the maximum profit among decisions \( D_1, \ldots, D_N \), i.e., \( Z^\text{pref} = \max_{1 \leq i \leq N} E[Z_{D_i}] \).
Viewing each $D_t$ as a bandit whose reward corresponds to the profit from selecting keywords in $D_t$, we can then formulate our problem as the multi-armed bandit problem. The multi-armed bandit problem has received much attention and many algorithms exist for this class of problems. Two such algorithms are the UCB1 (see [3]) and the EXP3 (see [4]) algorithms. Both algorithms adaptively select a bandit in each time period based on past observations. Let $(Z_t(UCB1) : t \geq 1)$ and $(Z_t(EXP3) : t \geq 1)$ denote the sequence of profits obtained by the algorithms UCB1 and EXP3, respectively. The average expected profit generated by these algorithms are given by (see [3, 4]):

$$1 \geq \frac{\sum_{t=1}^{T} Z_t(UCB1)}{T Z^{pref fix}} - \frac{1}{2} L (\alpha N \log T + \beta N)$$

$$1 \geq \frac{\sum_{t=1}^{T} Z_t(EXP3)}{T Z^{pref fix}} - \frac{1}{2} \frac{3}{2} \frac{\log N}{\beta N}$$

where $L \equiv \max \left\{ \sum_{i=1}^{N} \pi_i x_i \left| \sum_{i=1}^{N} c_i x_i \leq 1, x_i \in N \forall i \right. \right\}$ is an upper bound on the maximum possible profit in any one period and $\alpha N = \sum_{1 \leq i \leq N} 8/\Delta_i$, and $\beta N = (1 + \pi^2/3) \sum_{1 \leq i \leq N} \Delta_i$, and for any $1 \leq i \leq N \Delta_i = Z^{pref fix} - E[Z_{D_i}]$.

The above results show that, as $T$ increases to infinity, the average profit earned by the UCB1 and EXP3 algorithms converge to the maximum expected profit $Z^{pref fix}$ among $D_1, \ldots, D_N$. We know from Theorem 1 that $Z^{pref fix}$ provides a good approximation to $Z^*$, suggesting that the UCB1 and EXP3 algorithms may provide a viable algorithm for solving our problem. Unfortunately, these two generic algorithms for the multi-armed bandit problem ignore any special structure of the problem, and thus, the rates of convergence of these algorithms depend on the number of keywords $N$. As the number of keywords $N$ increases, the rate of convergence decreases as well, clearly an undesirable feature in our setting where we can have millions of keywords. In fact, the UCB1 algorithm requires us to try each of the $N$ decisions once during the first $N$ time periods. Clearly, for large values of $N$ as in our setting, this might not be feasible.

### 4.2 An Improved Adaptive Approximation Algorithm

Motivated by the result from the previous section, we will develop an alternative algorithm that exploits the special structure of our problem. We will show that our algorithm exhibits a performance bound that scales gracefully with the problem’s parameters and the bound will be independent of the number of keywords. Then, through extensive simulations in Section 5, we will show that our algorithm outperforms both the UCB1 and EXP3 algorithms.

Our algorithm, which we will refer to as the ADAPTIVE BIDDING algorithm, is defined as follows.

**ADAPTIVE BIDDING**

- **INITIALIZATION:**
  - For any $i$, let $y_i$ and $x_i$ denote the cumulative total number of impressions and the total number of clicks, respectively, that the ad associated with keyword $i$ has received. Set $y_i = x_i = 0$ for all $1 \leq i \leq N$. Set $\hat{p}_i^0 = 1$ for all $i$.
  - We choose a sequence $(\gamma_t \in [0,1] : t \geq 1)$, where $\gamma_t$ will denote the probability that we choose a random decision at time $t$. We will later suggest good choices for the $\gamma_t$ (see the end of this section and Appendix F).

- **DEFINITION:** For $t = 1, 2, \ldots$
  - Let $\ell_t$ be the index such that
    $$\ell_t \equiv \max \left\{ \ell : \mu \sum_{u=1}^{\ell} c_u \lambda_u p_u^{\ell} \leq 1 - \frac{1}{K} - \frac{2}{K(1-\alpha_{1/3})} \right\}$$
  - Let $\theta_t$ be an integer chosen uniformly at random from the set $\{1, 2, \ldots, N\}$
  - Let $F_t$ denote an independent binary random variable such that $\mathbb{P}\{F_t = 1\} = 1 - \gamma_t$ and $\mathbb{P}\{F_t = 0\} = \gamma_t$.
  - Let the decision $g_t$ be defined by
    $$g_t = \begin{cases} \ell_t, & \text{if } F_t = 1 \\ \theta_t, & \text{otherwise.} \end{cases}$$
  - Let $G_t = \{1, 2, \ldots, g_t\}$. Bid on keywords in $G_t$.
  - Observe the resulting impressions and clicks.
  - Updates: For any keyword $i$, let $V_i^t$ and $W_i^t$ denote the number of impressions and clicks that keyword $i$ receives in this period, respectively. Then, for all $i$,
    $$y_i = y_i + V_i^t \quad \text{and} \quad x_i = x_i + W_i^t$$
    $$\hat{p}_i^t = \begin{cases} 1, & \text{if } y_i = 0 \\ \frac{x_i}{y_i}, & \text{if } y_i > 0. \end{cases}$$

- **OUTPUT:** A sequence of decisions $(G_t : t \geq 1)$ and the corresponding sequence of indexes $(g_t : t \geq 1)$.

In the above algorithm, $\hat{p}_i^t$ represents our estimate, at the end of $t$ periods, of the clickthrough probability of the ad associated with keyword $i$. Note that as long as the ad associated with keyword $i$ has not received any impressions (i.e., we have no data on the clickthrough probability), we will set $\hat{p}_i^t$ to 1. However, when the ad receives at least one impressions, we set $\hat{p}_i^t$ as the average clickthrough probability.

For ease of exposition, let us introduce the following notation that will be used throughout the paper. Let $\mathcal{I}_U$ be defined by

$$\mathcal{I}_U = \max \left\{ \ell : \mu \sum_{u=1}^{\ell} c_u \lambda_u p_u \leq 1 - \frac{1}{K} - \frac{1}{K(1-m)/3} \right\}$$

Observe that $\mathcal{I}_U$ is the same as the index of the prefix we would use in the algorithm of Theorem 1 if we knew the probabilities $p_i$. We will show below that we can state the convergence of our algorithm in terms of $\mathcal{I}_U$ rather than $N$, the total number of keywords. Intuitively speaking, it does not make sense for us to choose too frequently prefixes whose index is larger than $\mathcal{I}_U$ since this will cause us to exceed our budget.

The main result of this section is stated in the following theorem.
Theorem 2. Let \((G_t : t \geq 1)\) denote the sequence of decisions generated by the ADAPTIVE BIDDING algorithm. Let \(\rho = E[\min \{S, \mu\}] / \mu\), and assume that \(1/k^{1-\alpha} \leq 1 - 1/k - 1/k^{(1-\alpha)/3}\). Then, under Assumption 1, for any \(T \geq 1\)

\[
\sum_{t=1}^{T} E[Z_{G_t}] / T \geq \rho \left( 1 - \frac{1}{k^{(1+2\alpha)/3}} + \frac{4}{(1-\alpha)^2/3} \right) - \frac{\sum_{t=1}^{T} \gamma_t}{T} - \frac{M}{T},
\]

where

\[
M = \frac{32\mu I_U}{k \rho (1 - \frac{1}{k^{(1+2\alpha)/3}})} + \frac{12I_{\lambda}^{2} \sum I_{i} \lambda / k}{k^{(7-4\alpha)/3}} \leq \frac{32\mu I_U}{k \rho (1 - \frac{1}{\sqrt{k}})} + \frac{12I_{\lambda}^{2} \sum I_{i} \lambda / k}{k} (1 - e^{-\lambda \rho}).
\]

Before we proceed to the proof of the theorem, let us compare the performance bound of our adaptive bidding algorithm with those obtained from the UCB1 and EXP3 algorithms for the multi-armed bandit problem. It follows from the discussion in Section 4.1 and from Theorem 1 that, under the algorithms UCB1 and EXP3, we have the following performance bounds: let \(\kappa \equiv \rho \left( 1 - \frac{1}{k^{(1+2\alpha)/3}} - 2/k^{(1-\alpha)/3} \right)\), then

\[
\sum_{t=1}^{T} Z_{i}(UCB1) / T Z^* \geq \kappa - L (\alpha \log T + \beta_N) / T Z^*
\]

\[
\sum_{t=1}^{T} Z_{i}(EXP3) / T Z^* \geq \kappa - 3 \sqrt{LN \log N / \sqrt{2T Z^*}}
\]

where \(L\) denotes the maximum possible profit in any one period.

When we compare the performance of UCB1 and EXP3 with the result of Theorem 2, we observe that there is a slightly decrease in the performance of our algorithm, from \(2/k^{(1-\alpha)/3}\) to \(4/k^{(1-\alpha)/3}\). However, the convergence rate of our algorithms is superior to both UCB1 and EXP3, as indicated below.

(a) Unlike its counterparts, the convergence rate of our algorithm in Theorem 2 is independent of the number of keywords \(N\). Instead, it depends only on \(I_U\) which reflects the maximum number of keywords (chosen based on a prefix ordering) whose expected cost does not exceed \(1 - 1/k - 1/k^{(1-\alpha)/3}\) fraction of the budget. In practice, \(I_U\) should be significantly smaller than \(N\).

(b) Although the convergence rate of our algorithm depends on the expected number of search queries \(\mu\) in any given period, this quantity should be comparable to the constant \(L\) – the maximum profit that can be obtained in any one period – appearing in the convergence rates of UCB1 and EXP3 above. To see this, note that by definition of \(L\), we have that

\[
L \geq E[Z_{i_1, ..., i_U}] \geq \rho \left( 1 - \frac{1}{k^{(1+2\alpha)/3}} \right) \sum_{t=1}^{I_U} \pi_i \lambda_i p_i,
\]

where the last inequality follows from the proof of Theorem 1. Moreover, the convergence rate of our algorithm depends on \(\mu\) through the product \(\mu I_U\). By definition, the value of \(I_U\) decreases as \(\mu\) increases, suggesting that the convergence rate of our algorithm should be insensitive to the increase in the parameter \(\mu\).

(c) With millions of available keywords, the arrival rate \(\lambda\mu\) of some keywords can be extremely small. The convergence rate of our algorithm, however, is insensitive to small arrival rates because the expression \(\lambda \mu / (1 - e^{-\lambda \rho})\) is bounded above by 1 as \(\lambda \mu\) converges to 0.

(d) We should also point out that, if we set \(\gamma_t = 1/t^2\) for all \(t \geq 1\), then \(\sum_{t=1}^{\infty} \gamma_t < \infty\). This implies that the average reward generated by our algorithm over \(T\) time periods converges to near-optimal profits at the rate of \(1/T\), which is faster than the convergence rate of other two algorithms.

(e) In their seminal paper, Lai and Robbins [16] established a lower bound on the minimum regret of any adaptive strategy, showing that for any non-anticipatory strategy \(\phi\),

\[
\sum_{t=1}^{T} \{Z^* - E[Z_t(\phi)]\} \geq \Omega(\log T),
\]

where \(E[Z_t(\phi)]\) denote the expected profit in period \(t\) under the strategy \(\phi\). This result shows that the total expected profit over \(T\) periods of any adaptive strategy must differ from the optimal expected profit by at least \(\Omega(\log T)\); thus, the average expected reward per time period cannot converge to the optimal expected profit faster than \(O(\log T) / T\).

We should note that our result in Theorem 2 is consistent with this lower bound. According to Theorem 2, we have that

\[
\sum_{t=1}^{T} \{Z^* - E[Z_{G_t}]\} \leq \nu Z^* T + \sum_{t=1}^{T} \gamma_t + M,
\]

where \(\nu = 1 - \rho \left( 1 - \frac{1}{k^{(1+2\alpha)/3}} - 2/k^{(1-\alpha)/3} \right)\). Clearly, the upper bound on the right hand side of the above inequality increases linearly with \(T\). Intuitively speaking, the average expected profit under our algorithm converges to the optimal expected profit under the prefix ordering, which serves only as an approximation to the optimal expected profit (from among all subsets of keywords).

Finally, in addition to nicer theoretical properties, in Section 5, through simulations on many large problem instances, we will further show that our adaptive bidding algorithm outperforms both UCB1 and EXP3, increasing the profits by an average of 7%.

4.3 Sketch of the Proof of Theorem 2

We provide a sketch of the proof of Theorem 2. For more details, the reader is referred to Appendix G. In Lemma 8 in Appendix G, we establish an upper bound on the probability that \(\ell_t > I_U\), showing that for all \(t \geq 1\), \(P\{\ell_t > I_U\} \leq 1/k^{(1-\alpha)/3}\). For any \(T \geq 1\), let the random variable \(J_T \subseteq \{1, 2, ..., T\}\) denote the set of time periods when the index of the last keyword chosen based on the prefix ordering is less than or equal to \(I_U\), i.e., \(J_T = \{t \leq T : \ell_t \leq I_U\}\). Note that when \(F_t = 1\), the algorithm chooses the decision based on the prefix ordering, i.e., \(q_t = \ell_t\). The result from Lemma 8 shows that \(J_T\) will contain a large fraction of time periods, enabling us to restrict our attention to only time periods in \(J_T\).

Let \(\xi = 1 - 1/k^{(1+2\alpha)/3}\). For any keyword \(i\), let \(\zeta_{i,T} = E[\sum_{t \in J_T, g_t \geq i} [\hat{p}_t - \hat{p}_t^{-1}]]\) denote the expected cumulative
errors during time periods in $J_T$ between our estimate of the clickthru probability for keyword $i$ and its true value. Lemma 9 (in Appendix G) then relates the performance of our algorithm to the accuracy of our estimates of the clickthru probabilities associated with keywords $1, \ldots, I_U$ during time periods in $J_T$, showing that for any $T \geq 1$,

$$\frac{\sum_{t=1}^{T} X_{i,t}}{T} \geq p \left( 1 - \frac{3}{k(1-\alpha)^3} \right) - \frac{\sum_{t=1}^{T} \gamma_i}{T} - \rho \mu \frac{\sum_{t=1}^{T} c_i \lambda_i \gamma_i}{T}.$$

In Lemma 10 in Appendix G, we then show that, on average, the cumulative errors over time periods in $J_T$ do not increase with $T$. In fact, it is bounded by a small constant plus a term that vanishes to zero as $T$ increases, i.e. for any $T \geq 1$ and $0 < \delta < 1$,

$$\mu \frac{\sum_{t=1}^{T} c_i \lambda_i \gamma_i}{T} \leq \frac{1}{k(1-\alpha)^3} + \frac{8\mu Z_U}{\delta^2 k^2 \xi^3} + \frac{6Z_U^2 \sum_{t=1}^{T} \gamma_i}{k^2(7-4\alpha)\lambda_i (1 - \delta) p^2 \rho}.$$

The result of Theorem 2 follows from the last two inequalities and from setting $\delta = 1/2$. The intuition underlying the above inequality is the following. Consider any keyword $i \leq I_U$. Suppose $J_T$ consists of $t_1 \leq \ldots \leq t_w$ and consider any $t_{w+1} \in J_T$. During the previous $h$ periods $t_1, \ldots, t_w$, the total expected number of queries associated with keyword $i$ is $\lambda_i \mu h$. By the definition of $J_T$, the prefix chosen during these $h$ periods has final index at most $I_U$. We can thus apply Lemma 5 to show that with probability at least $\xi$ we have enough budget to display ads for each of these $\lambda_i \mu h$ queries of keyword $i$. Hence, the expected cumulative number of impressions that the ad for keyword $i$ will receive just before period $t_{w+1}$ is at least $O(\lambda_i \mu h)$. We can then apply a Chernoff bound to show that, at time $t_{w+1}$, the error between our empirical estimate of the probability and its true value is at most $O(e^{-c(\lambda_i \mu h)})$, declining to zero exponentially in $h$. Thus, the total errors across all time periods in $J_T$ (summing over $h$) will be finite and independent of $T$.

5. EXPERIMENTS

In the previous section, we developed an alternative algorithm that exploits the prefix ordering of the keywords. We showed that the average expected profit generated by the algorithm converges to near-optimal profits and the convergence rate is independent of the number of keywords. In this section, we compare the performance of the algorithm against UCB1 and EXP3 algorithms for multi-armed bandit problems.

We ran two experiments: SMALL and LARGE. Each experiment has 100 randomly generated problem instances. In the SMALL experiment, each problem instance has 8,000 keywords, 200 time periods, $400 budget per period, and the number of search queries in each time period has a Poisson distribution with a mean of 40,000. In the LARGE experiment, each problem instance has 50,000 keywords, 200 time periods, $1,000 budget per period, and the number of search queries in each time period has a Poisson distribution with a mean of 150,000.

In both sets of experiments, for each problem instance, the cost-per-click of each keyword is chosen uniformly at random from the interval $[0,1.0,3]$ and the profit-per-click of each keyword is chosen uniformly at random from the interval $[0,1.0,3]$. The clickthru rate associated with each keyword is a random number from $[0.0,0.2]$. To generate the probability that a search query will correspond to each keyword, we assign random numbers (chosen independently and uniformly from $[0,1]$) to all keywords and normalize them.

For each problem instance, we compare our algorithm with EXP32 and UCB12, two standard algorithms for the multi-armed bandit problem (see Section 4.1 for more details). For our prefix-based algorithm, we set the randomization probability $\gamma_i$ to be $\gamma_i = 1/t^2$ for all $t \geq 1$.

Figure 2(a) and (b) shows the simulation results for the SMALL experiment. Figure 2(a) shows the average profit over time for all three algorithms when averaged over all 100 problem instances in the SMALL experiment. The dashed lines above and below the solid lines correspond to the 95% confidence interval. As seen from the figure, our prefix-based algorithm outperforms both UCB1 and EXP3 algorithms, yielding an increase of about 7% in average profits. Figure 2(b) shows the distribution over 100 problem instances (in the SMALL experiment) of the relative difference between the linear programming upper bound (Lemma 3) and the profit under our algorithm. As seen from the figure, the profit generated by our algorithm differs from the linear programming bound by about 30%. Since the optimal solution of the linear program provides an upper bound on the optimal expected profit, this result implies that the average profit generated by our algorithm differs from the optimal by at most 30%.

Note that for the SMALL experiments, the value of $k$ is about 1200 and the value of $\alpha$ is around 0.33 (see Assumption 1 for more details). The bound of Theorem 2 implies that for these values our algorithm should converge to at least 16 of the linear programming bound, but our simulation results show that our algorithm performs much better than this. As indicated by Theorems 1 and 2, as the value of $k$ increases and the value of $\alpha$ decreases, the average profit should approach the optimal. This theorem is confirmed by the simulation results for the LARGE experiments, which is shown in Figure 2(c) and (d). For the LARGE experiment, the value of $k$ and $\alpha$ are around 3,333 and 0.22, respectively. For these values, the bound of Theorem 2 implies that our algorithm should converge to at least 61 of the linear programming bound, and again in our simulation we are doing better than this. In Figure 2(c), we see that the average profit generated by our algorithms is about 20% higher than the profits under UCB1 and EXP3 algorithms. In addition, as shown in Figure 2(d), the average profit differs from the optimal by at most 20%.

Appendix F provides additional discussion of the impact of different choices of randomization parameters $\gamma_i$ and initialization of the estimated clickthru probabilities.

6. MODEL DISCUSSIONS AND EXTENSIONS

In this section, we provide additional discussion on the strengths and weaknesses of our formulation, along with possible extensions of our model. First, in most search-based advertising services, changing the bid price of each keyword will alter the position of the ad on the search result page.
This work focuses primarily on identifying profitable subsets of keywords. Our formulation assumes that the bid price of each keyword remains constant and the ad will always appear in the same spot on search result pages. Incorporating the impact of the ad's position on the search result page will allow for a richer class of models. Finding an appropriate model that enables the development of efficient computational methods remains an open question. Another open question is that of finding an appropriate model for competitive bidding for keywords. Our model assumes that the cost of a keyword is fixed. Of course, these costs depend on what others bid for the same keywords. Incorporating competitive aspects into our model is an interesting and challenging research direction.

7. REFERENCES


Figure 2: Figures in the top ((a) and (b)) and bottom ((c) and (d)) rows show the result for the SMALL and LARGE experiments, respectively. The figures in the left hand column ((a) and (c)) show a comparison of the average profit over time under UCB1, EXP3, and our prefix-based algorithms. The figures on the right hand column ((b) and (d)) show the relative difference between the linear programming bound and the average profit generated by our algorithm over 100 problem instances.
APPENDIX

A. FIGURES

Figure 3: Examples of ads shown under search-based advertising programs. The left and right figures show the result of searching for “cheap Europe trip” (on 10/18/05) on Google and Yahoo!, respectively. The links on the top (shaded) area and on the right hand column of the search result page correspond to ads by companies participating in the search-based advertising programs.

Figure 4: Figure (a) is a screen shot showing the estimated number of search queries in Yahoo! in September 2005 related to the keyword “cheap Europe trip”. The data is available through the Keyword Selector Tool offered by Yahoo! at http://inventory.overture.com/d/searchinventory/suggestion/. Figure (b) shows the current bids (as of 10/18/05) for the same keyword under the Yahoo! Sponsored Marketing program. This information is publicly available through the Bid Tools offered by Yahoo! at http://uv.bidtool.overture.com/d/search/tools/bidtool/index.jhtml.

B. PROOF OF LEMMA 1

Proof. Recall that \( B^r \) denotes the remaining account balance when the \( r^{th} \) search query arrives and we bid on keywords in \( A \). This implies that the event \( \{ B^r \geq c_i \} \) depends only on the previous \( r - 1 \) search queries, and does not depend on whether the consumer clicks on the ad associated with keyword \( i \) when the \( r^{th} \) search query arrives nor does it depend on
Figure 5: A distribution of the median bid price under the Yahoo! Sponsored Search program for 842 travel-related keywords collected during a 2-day period from 10/17/05 - 10/18/05. A complete list of 842 keywords used in this study can be found at http://www.orie.cornell.edu/~paatrus/keywordlist.txt.

Figure 6: A distribution of average daily searches on Yahoo! for 824 travel-related keywords during September 2005.

where the third equality follows from our assumption that the clickthroughs of the ads are independent of the keyword corresponding to the \( r \)th search query, and both of these random variables are independent of the total number of arrivals \( S \). The second equality in the lemma follows directly from the above result. 

C. PROOF OF LEMMA 2

The proof of Lemma 2 makes use of the following bound on the tail of a Poisson random variable. Since the proof follows from the standard application of Chernoff bound (cf. Theorem 4.5 in [20]), we omit the details.

**Lemma 6.** If \( S \) is a Poisson random variable with mean \( \mu \), then for any \( 0 < \delta < 1 \), \( P \{ S \leq (1 - \delta)\mu \} \leq e^{-\mu\delta^2/2} \).

Here is the proof of Lemma 2.
Proof. We have that
\[
\sum_{r=1}^{[\mu]} \mathbb{P}\{ S \geq r \} = \sum_{r=1}^{[\mu]} (1 - \mathbb{P}\{ S < r \}) = [\mu] - \sum_{r=1}^{[\mu]} \mathbb{P}\{ S \leq r - 1 \}
\]
It follows from Lemma 6 that for any \(1 \leq r \leq [\mu]\),
\[
\mathbb{P}\{ S \leq r - 1 \} = \mathbb{P}\left\{ S \leq \frac{r - 1}{\mu} \right\} \leq \exp \left\{ -\mu \left( 1 - \left( \frac{r - 1}{\mu} \right) \right)^2 / 2 \right\} = e^{-\left( (1+\mu-r)^2/2\mu \right)} \leq e^{-\left( (1+|\mu|-r)^2/2\mu \right)},
\]
which implies that
\[
\sum_{r=1}^{[\mu]} \mathbb{P}\{ S \leq r - 1 \} \leq \sum_{r=1}^{[\mu]} e^{-\left( (1+|\mu|-r)^2/2\mu \right)} = \sum_{r=1}^{[\mu]} e^{-r^2/2\mu} \leq \int_0^{\infty} e^{-x^2/2\mu} dx = \frac{\sqrt{2\pi\mu}}{2}
\]
Therefore, we have that
\[
\sum_{r=1}^{[\mu]} \mathbb{P}\{ S \geq r \} \geq [\mu] - \frac{\sqrt{2\pi\mu}}{2} = \mu \left( \frac{[\mu]}{\mu} - \frac{\sqrt{2\pi}}{2\sqrt{\mu}} \right) \geq \mu \left( 1 - \frac{1}{\mu} - \frac{\sqrt{2\pi}}{2\sqrt{\mu}} \right) \geq \mu \left( 1 - \frac{3}{\sqrt{\mu}} \right),
\]
where the last inequality follows from our assumption that \(\mu > 1\), which implies that \(1/\sqrt{\mu} > 1/\mu\) and the fact that \(\sqrt{2\pi}/2 \leq 2\).

D. PROOF OF LEMMA 4

Proof. Since the second inequality follows directly from Lemma 3, it suffices to prove only the first inequality. Let \(m\) denote the index such that \(\mu \sum_{i=1}^{m} c_i \lambda_i p_i \leq 1 < \mu \sum_{i=1}^{m+1} c_i \lambda_i p_i\). Since \(\pi_1/c_1 \geq \pi_2/c_2 \geq \cdots \geq \pi_n/c_n\), it follows from the definition of \(Z^{LP}\) that
\[
Z^{LP} = \mu \sum_{i=1}^{m} \pi_i \lambda_i p_i + \mu \sum_{i=m+1}^{n} \lambda_i p_i \left( \frac{1 - \mu \sum_{i=1}^{m} c_i \lambda_i p_i}{c_{m+1} \lambda_{m+1} p_{m+1}} \right)
\]
It is easy to verify that \(u < m\) since \(c_i \lambda_i p_i \mu \leq 1/k^{1-a} \leq 1/k(1-a)/3\) for all \(i\) by Assumption 1 (c). Moreover, since \(\pi_1/c_1 \geq \cdots \geq \pi_N/c_N\), we have that
\[
\sum_{i=u+1}^{m} \pi_i \lambda_i p_i \leq \left( \sum_{i=1}^{u} \pi_i \lambda_i p_i \right) \frac{\sum_{i=u+1}^{m} c_i \lambda_i p_i}{\sum_{i=1}^{u} c_i \lambda_i p_i} \quad \text{and} \quad \sum_{i=u+1}^{m} \pi_i \lambda_i p_i \leq \left( \sum_{i=1}^{u} \pi_i \lambda_i p_i \right) \left( \frac{c_{m+1} \lambda_{m+1} p_{m+1}}{\sum_{i=1}^{u} c_i \lambda_i p_i} \right)
\]
which implies that
\[
\sum_{i=u+1}^{m} \pi_i \lambda_i p_i \leq \left( \sum_{i=1}^{u} \pi_i \lambda_i p_i \right) \left( \frac{\sum_{i=u+1}^{m} c_i \lambda_i p_i}{\sum_{i=1}^{u} c_i \lambda_i p_i} \right) \quad \text{and} \quad \pi_{m+1} \lambda_{m+1} p_{m+1} \leq \left( \sum_{i=1}^{u} \pi_i \lambda_i p_i \right) \left( \frac{c_{m+1} \lambda_{m+1} p_{m+1}}{\sum_{i=1}^{u} c_i \lambda_i p_i} \right).
\]
Putting everything together, we have
\[
Z^{LP} = \mu \sum_{i=1}^{u} \pi_i \lambda_i p_i + \mu \sum_{i=u+1}^{m} \pi_i \lambda_i p_i + \mu \sum_{i=m+1}^{n} \lambda_i p_i \left( \frac{1 - \mu \sum_{i=1}^{m} c_i \lambda_i p_i}{c_{m+1} \lambda_{m+1} p_{m+1}} \right)
\]
\[
\leq \mu \sum_{i=1}^{u} \pi_i \lambda_i p_i \left( 1 + \frac{\sum_{i=u+1}^{m} c_i \lambda_i p_i}{\sum_{i=1}^{u} c_i \lambda_i p_i} + \frac{1}{\mu} \left( \frac{c_{m+1} \lambda_{m+1} p_{m+1}}{\sum_{i=1}^{u} c_i \lambda_i p_i} \right) \right)
\]
\[
\leq \mu \sum_{i=1}^{u} \pi_i \lambda_i p_i \left( 1 + \frac{\sum_{i=u+1}^{m} c_i \lambda_i p_i}{\sum_{i=1}^{u} c_i \lambda_i p_i} + \frac{1 - \mu \sum_{i=1}^{m} c_i \lambda_i p_i}{\mu \sum_{i=1}^{u} c_i \lambda_i p_i} \right)
\]
which is the desired result.

E. PROOF OF LEMMA 5

The proof of Lemma 5 makes use of the following result.

Lemma 7. For any \(A \subseteq \{1, 2, \ldots, N\}\),
\[
\text{Var} \left( \sum_{i=1}^{r-1} \sum_{i \in A} c_i 1 \left( Q_s = i, X_{si} = 1 \right) \right) \leq \frac{r-1}{k} \sum_{i \in A} c_i \lambda_i p_i
\]
PROOF. Since $Q_s$’s are independent and for any $i$, $X_{si}$’s are also independent, it suffices to show that for any $s < r$,

$$\text{Var} \left( \sum_{i \in A} c_i 1(Q_s = i, X_{si} = 1) \big| S \geq r \right) \leq \frac{1}{k} \sum_{i \in A} c_i \lambda_i p_i$$

Note that since $Q_s$ and $X_{si}$ are independent, we have that $E \left[ 1(Q_s = i, X_{si} = 1) \big| S \geq r \right] = \lambda_i p_i$, which implies that

$$\text{Var} \left( \sum_{i \in A} c_i 1(Q_s = i, X_{si} = 1) \big| S \geq r \right) = E \left[ \left( \sum_{i \in A} c_i (1(Q_s = i, X_{si} = 1) - p_i \lambda_i) \right)^2 \big| S \geq r \right]$$

$$= \sum_{i \in A} c_i^2 E \left[ (1(Q_s = i, X_{si} = 1) - p_i \lambda_i)^2 \big| S \geq r \right]$$

$$+ \sum_{i,j \in A \neq j} c_i c_j E \left[ (1(Q_s = i, X_{si} = 1) - \lambda_i, p_i) \cdot (1(Q_s = j, X_{sj} = 1) - \lambda_j, p_j) \big| S \geq r \right]$$

$$= \sum_{i \in A} c_i^2 \lambda_i (1 - \lambda_i, p_i) - \sum_{i,j \in A \neq j} c_i c_j \lambda_i \lambda_j p_i$$

$$\leq \frac{1}{k} \sum_{i \in A} c_i \lambda_i p_i,$$

where the third equality follows from the fact that $1(Q_s = i)1(Q_s = j) = 0$ for all $i \neq j$. The final inequality follows from Assumption 1(c) that $c_i \leq 1/k$ for all $i$. \qed

Here is the proof of Lemma 5.

PROOF. For any keyword $i \in H$, let $T_r^H$ denote the total amount of money that we have already spent when $r^{th}$ search query arrives. Then,

$$\mathcal{P} \left\{ B_r^H \geq c_i \big| S \geq r \right\} = \mathcal{P} \left\{ T_r^H \leq 1 - c_i \big| S \geq r \right\} = 1 - \mathcal{P} \left\{ T_r^H > 1 - c_i \big| S \geq r \right\}.$$ 

We will focus on developing an upper bound for the expression $\mathcal{P} \left\{ T_r^H > 1 - c_i \big| S \geq r \right\}$.

By definition of $T_r^H$, we know that, with probability 1,

$$T_r^H = \sum_{s=1}^{r-1} \sum_{i \in H} c_i 1(Q_s = i, B_r^H \geq c_i, X_{si} = 1) \leq \sum_{s=1}^{r-1} \sum_{i \in H} c_i 1(Q_s = i, X_{si} = 1).$$

Moreover, by the hypothesis of the lemma,

$$E \left[ \sum_{s=1}^{r-1} \sum_{i \in H} c_i 1(Q_s = i, X_{si} = 1) \big| S \geq r \right] = (r - 1) \sum_{i \in H} c_i \lambda_i p_i \leq \mu \sum_{i \in H} c_i \lambda_i p_i \leq 1 - \frac{1}{k} - \frac{1}{k^{(1-\alpha)/3}},$$

and it follows from the Chebyshev’s inequality and the previous lemma that

$$\mathcal{P} \left\{ T_r^H > 1 - c_i \big| S \geq r \right\} \leq \mathcal{P} \left( \sum_{s=1}^{r-1} \sum_{i \in H} c_i 1(Q_s = i, X_{si} = 1) > 1 - c_i \big| S \geq r \right)$$

$$\leq \frac{\text{Var} \left( \sum_{s=1}^{r-1} \sum_{i \in H} c_i 1(Q_s = i, X_{si} = 1) \big| S \geq r \right)}{(1 - c_i - (r - 1) \sum_{i \in H} c_i (p_i, \lambda_i))^2}$$

$$\leq \frac{\frac{1}{k^{(2-2\alpha)/3}} \sum_{i \in H} c_i \lambda_i p_i}{(1 - c_i - (r - 1) \sum_{i \in H} c_i (p_i, \lambda_i))^2}$$

$$\leq \frac{\frac{1}{k^{(2-2\alpha)/3}} \mu \sum_{i \in H} c_i \lambda_i p_i}{(1 - c_i - (r - 1) \sum_{i \in H} c_i (p_i, \lambda_i))^2}$$

$$\leq \frac{1}{k^{(1+2\alpha)/3}}$$
which is the desired result.

F. IMPACT OF RANDOMIZATION AND INITIALIZATION

In this section, we explore the impact of different choices of randomization and initialization. We ran our algorithm on the 100 problem instances in the LARGE experiment using four different randomization and initialization settings:

(a) No randomization \((\gamma_t = 0 \ \forall t)\) and set the initial estimate of the clickthru probability associated with the ad of each keyword to zero \((\hat{p}_0^i = 0 \ \forall i)\).

(b) \(1/t\) randomization \((\gamma_t = 1/t \ \forall t)\) and set the initial estimate of the clickthru probability associated with the ad of each keyword to 1 \((\hat{p}_0^i = 1 \ \forall i)\).

(c) \(1/t^2\) randomization \((\gamma_t = 1/t^2 \ \forall t)\) and set the initial estimate of the clickthru probability associated with the ad of each keyword to 1 \((\hat{p}_0^i = 1 \ \forall i)\).

(d) No randomization \((\gamma_t = 0 \ \forall t)\) and set the initial estimate of the clickthru probability associated with the ad of each keyword to 1 \((\hat{p}_0^i = 1 \ \forall i)\).

Figure 7 shows the average profit over time under each of the four parameter settings. Recall that our ADAPTIVE ALGORITHM sets the initial estimate of the clickthru probability to 1, corresponding to cases b), c), and d). In this case, we clearly see the benefits of randomization; allowing for random decisions with small probability \((\gamma_t = 1/t \text{ or } 1/t^2)\) yields significantly higher average profits. It is interesting to note that although setting \(\gamma_t = 1/t\) yields slightly higher profits over \(\gamma_t = 1/t^2\), the difference is quite small.

By setting the initial clickthru probability for each keyword to 1, we limit the number of keywords chosen in each time period. Initially, we will select a small subset of keywords and gradually enlarge the set of keywords as we observe more data on impressions and clicks. The opposite extreme of this approach is to set the initial clickthru probability for each keyword to 0, implying that we will select all keywords in the first time period. Although this setting yields the highest average profit, as seen from the top line in Figure 7, we do not have a formal proof of the convergence under this setting.

G. PROOF OF THEOREM 2

The proof of Theorem 2 depends on a series of lemmas. The first lemma establishes an upper bound on the probability that \(\ell_t > I_U\). The proof appears in Appendix G.1.

**Lemma 8.** For any \(t \geq 1\), \(P(\ell_t > I_U) \leq 1/k^{(1-\alpha)/2}\).
The next lemma establishes a lower bound on the average total profit. The proof of this result appears in Appendix G.2.

**Lemma 9.** For any $T \geq 1$,

$$\frac{\sum_{t=1}^{T} E[Z_{G_t}]}{TZ^*} \geq \rho \left( 1 - \frac{1}{k(1+2\alpha)/\beta} - \frac{3}{k(1-\alpha)/\beta} \right) \geq \frac{\sum_{t=1}^{T} \gamma_t}{T} - \rho \left( 1 - \frac{1}{k(1+2\alpha)/\beta} \right) \sum_{t=1}^{T} c_i \lambda_i E \left[ \sum_{t \geq 1} \gamma_t \right] \left[ 1_{g_t, 1(F_t=1, t \leq T_U) \geq i} \right]$$

We should note that if $J_T = \{ t \leq T : F_t = 1, t_i \leq T_U \}$, then since the keywords are indexed starting from 1, we have that

$$\sum_{t \in J_T} 1_{(F_t=1, t_i \leq T_U) \geq i} \geq \sum_{t \in J_T} 1_{g_t \geq i} \geq 1$$

which consistent with the notations in Section 4.3.

The next result establishes an upper bound on the accumulated difference between the empirical estimate of the clickthru probability $\hat{p}^{t-1}_i$ and its true value $p_i$, corresponding to the second term in the above lemma. The proof appears in Appendix G.3.

**Lemma 10.** For any $T \geq 1$ and $0 < \delta < 1$,

$$\mu \sum_{i=1}^{T_U} c_i \lambda_i E \left[ \sum_{t \leq T \gamma_t, 1(F_t=1, t_i \leq T_U) \geq i} |p_i - \hat{p}^{t-1}_i| \right] \leq \frac{T}{k(1-\alpha)/\beta} - \frac{8\mu T_U}{\delta^2 k^2 \lambda^2 \left( 1 - \frac{1}{\kappa(1+2\alpha)/\beta} \right)^2} + \frac{6\mu T_U}{k(7-4\alpha)/3(1-\delta)} \sum_{i=1}^{T_U} \lambda_i \mu / (1 - e^{-\lambda_i}) \rho \left( 1 - \frac{1}{\kappa(1+2\alpha)/\beta} \right)$$

Finally, here is the proof of Theorem 2.

**Proof.** It follows from Lemma 9 and 10 that, for any $0 < \delta < 1$,

$$\frac{\sum_{t=1}^{T} E[Z_{G_t}]}{TZ^*} \geq \rho \left( 1 - \frac{1}{k(1+2\alpha)/\beta} - \frac{4}{k(1-\alpha)/\beta} \right) \frac{\sum_{t=1}^{T} \gamma_t}{T} \geq 0$$

Setting $\delta = 1/2$, we get that

$$\frac{\sum_{t=1}^{T} E[Z_{G_t}]}{TZ^*} \geq \rho \left( 1 - \frac{1}{k(1+2\alpha)/\beta} - \frac{4}{k(1-\alpha)/\beta} \right) - \frac{\sum_{t=1}^{T} \gamma_t}{T} - M$$

where

$$M = \frac{32\mu T_U}{k\rho \left( 1 - \frac{1}{\kappa(1+2\alpha)/\beta} \right)} + \frac{12\mu T_U \sum_{i=1}^{T_U} \lambda_i \mu / (1 - e^{-\lambda_i})}{k(7-4\alpha)/3(1-\delta)} \leq \frac{32\mu T_U}{k\rho \left( 1 - \frac{1}{\kappa(1+2\alpha)/\beta} \right)} + \frac{12\mu T_U \sum_{i=1}^{T_U} \lambda_i \mu / (1 - e^{-\lambda_i})}{k}$$

where the inequality follows from the fact that $0 \leq \alpha < 1$, which implies that $1 - 1/k(1+2\alpha)/\beta \geq 1 - 1/\sqrt{k}$ and $k(7-4\alpha)/3 \geq k$, which is the desired result. \( \Box \)

### G.1 Proof of Lemma 8

**Proof.** For any keyword $u$, let $Y_u(t)$ denote the *cumulative* total number of impressions that the ad associated with
Therefore, we have that

\[
X_u \leq \frac{2}{k^{(1-\alpha)/3}} \sum_{u=1}^{I_u+1} c_u \lambda_u p_u - 1 - \frac{1}{k^{(1-\alpha)/3}} \mu \sum_{u=1}^{I_u+1} c_u \lambda_u p_u
\]

where the first inequality follows from the definition of \(I_u\), which implies that \(1 - 1/k - 1/k^{(1-\alpha)/3} < \mu \sum_{u=1}^{I_u+1} c_u \lambda_u p_u\). Therefore, we have that

\[
\mu \sum_{u=1}^{I_u+1} c_u \lambda_u p_u - \left( 1 - \frac{1}{k} - \frac{2}{k^{(1-\alpha)/3}} \right) > \frac{1}{k^{(1-\alpha)/3}}.
\]

The final inequality follows from the fact that for any keyword \(u\) such that \(Y_u(t-1) = 0\), we have \(\hat{p}_u^{(t-1)} = 1\), which implies that \(p_u - \hat{p}_u^{(t-1)} \leq 0\).

By Markov’s inequality, we have the following inequalities

\[
\mathbb{P} \left\{ \mu \sum_{u=1}^{I_u+1} c_u \lambda_u \left( p_u - \hat{p}_u^{(t-1)} \right) 1 (Y_u(t-1) \geq 1) > \frac{1}{k^{(1-\alpha)/3}} \right\}
\]

\[
\leq \frac{E \left( \left( \mu \sum_{u=1}^{I_u+1} c_u \lambda_u \left( p_u - \hat{p}_u^{(t-1)} \right) 1 (Y_u(t-1) \geq 1) \right)^2 \right)}{(1/k^{(1-\alpha)/3})^2}
\]

\[
= k^{(2-2\alpha)/3} \sum_{u=1}^{I_u+1} (\mu c_u \lambda_u)^2 E \left( (p_u - \hat{p}_u^{(t-1)})^2 1 (Y_u(t-1) \geq 1) \right)
\]

\[
+ 2k^{(2-2\alpha)/3} \sum_{u<v} \mu^2 c_u c_v \lambda_u \lambda_v E \left( (p_u - \hat{p}_u^{(t-1)}) (p_v - \hat{p}_v^{(t-1)}) 1 (Y_u(t-1) \geq 1, Y_v(t-1) \geq 1) \right)
\]

Now, consider any \(1 \leq u < v \leq I_u + 1\),

\[
E \left[ (p_u - \hat{p}_u^{(t-1)}) (p_v - \hat{p}_v^{(t-1)}) 1 (Y_u(t-1) \geq 1, Y_v(t-1) \geq 1) \right]
\]

\[
= \sum_{y_u \geq 1, y_v \geq 1} \mathbb{P} \{ Y_u(t-1) = y_u, Y_v(t-1) = y_v \} E \left[ (p_u - \hat{p}_u^{(t-1)}) (p_v - \hat{p}_v^{(t-1)}) \mid Y_u(t-1) = y_u, Y_v(t-1) = y_v \right]
\]

\[
= \sum_{y_u \geq 1, y_v \geq 1} \mathbb{P} \{ Y_u(t-1) = y_u, Y_v(t-1) = y_v \} E \left[ \left( \frac{1}{y_u} \sum_{s=1}^{y_u} X_u^s - p_u \right) \left( \frac{1}{y_v} \sum_{s=1}^{y_v} (X_v^s - p_v) \right) \mid Y_u(t-1) = y_u, Y_v(t-1) = y_v \right]
\]

\[
= 0,
\]

where \(X_u^s\) is an indicator random variable indicating whether or not the consumer clicks on the ad associated with keyword \(u\) during the \(s^{th}\) time when the ad associated with keyword \(u\) receives an impression. The final equality follows from the fact \(X_u^s\) and \(X_v^s\) are independent of \(Y_u(t)\) and \(Y_v(t)\). Moreover, \(X_u^s\) and \(X_v^s\) are also independent for any \(s\) and \(t\) and \(E [X_u^s - p_u] = E [X_v^s - p_v] = 0\) for all \(s\).
In addition,

\[
E \left[ (p_u - \hat p_u^{i-1})^2 1(Y_u(t-1) \geq 1) \right] \\
= \sum_{y_u \geq 1} \mathcal{P} \{Y_u(t-1) = y_u\} E \left[ (p_u - \hat p_u^{i-1})^2 \left| Y_u(t-1) = y_u \right. \right] \\
= \sum_{y_u \geq 1} \mathcal{P} \{Y_u(t-1) = y_u\} E \left[ \left( \frac{1}{y_u} \sum_{s=1}^{y_u} (X_u^s - p_u) \right) \left( \frac{1}{y_u} \sum_{s=1}^{y_u} (X_u^s - p_u) \right) \left| Y_u(t-1) = y_u \right. \right] \\
= \sum_{y_u \geq 1} \mathcal{P} \{Y_u(t-1) = y_u\} E \left[ \frac{1}{y_u^2} \sum_{s=1}^{y_u} (X_u^s - p_u)^2 \left| Y_u(t-1) = y_u \right. \right] \\
= \sum_{y_u \geq 1} \mathcal{P} \{Y_u(t-1) = y_u\} \frac{p_u(1 - p_u)}{y_u} \\
\leq p_u
\]

where the third equality follows from the fact that $X_u^s$’s are independent and identically distributed Bernoulli random variables with parameter $p_u$ and they are independent of $Y_u(t)$. Moreover, $E[X_u^s - p_u] = 0$ for all $s$. The fourth equality follows from the fact that $X_u^s$ are Bernoulli random variable with parameter $p_u$.

Putting everything together, we have that

\[
\mathcal{P} \{\ell_t > I_U\} \leq \mathcal{P} \left\{ \mu \sum_{u=1}^{I_{U+1}} c_u \lambda_u \left( p_u - \hat p_u^{i-1} \right) 1(Y_u(t-1) \geq 1) > \frac{1}{k(1-\alpha)^3} \right\} \\
\leq k^{(2-2\alpha)/3} \sum_{u=1}^{I_{U+1}} (\mu c_u \lambda_u)^2 p_u \\
\leq k^{(2-2\alpha)/3} \frac{1}{k^{1-\alpha}} \sum_{u=1}^{I_{U+1}} \mu c_u \lambda_u p_u \\
= \frac{1}{k(1-\alpha)^3} \mu \sum_{u=1}^{I_{U+1}} c_u \lambda_u p_u \\
\leq \frac{1}{k(1-\alpha)^3}
\]

where the third inequality follows from Assumption 1(c), which implies that, for any keyword $i$, $\mu c_i \lambda_i \leq 1/k^{1-\alpha}$. The final inequality follows from the definition of $I_U$, which implies that

\[
\mu \sum_{u=1}^{I_{U+1}} c_u \lambda_u p_u \leq 1 - \frac{1}{k} - \frac{1}{k(1-\alpha)^3} + \frac{1}{k^{1-\alpha}} \leq 1
\]

\[
\square
\]

### G.2 Proof of Lemma 9

**Proof.** Recall that, for any $t \geq 1$, we have $G_t = \{1, \ldots, g_t\}$. Also, let $S^t$ denote the total number of search queries in time $t$, and let $B_0^{(t)}$ denote the remaining account balance in period $t$ just before the arrival of the $r^{th}$ search query associated with keyword $i$ (in period $t$), assuming that we have bid on the set of keywords in $\{1, 2, \ldots, g_t\}$ in period $t$. Also, let the random variable $Q_{r,t}$ denote the keyword associated with the $r^{th}$ search query in period $t$. Also, let $X_{r,t}$ denote a Bernoulli random variable indicating whether or not the consumer clicks on the ad associated with keyword $i$ during the arrival of the $r^{th}$ search query in period $t$. Then, we have that
\[
\sum_{t=1}^{T} E[Z_{G_t}] = \sum_{t=1}^{T} E \left[ \sum_{r=1}^{S_t} \sum_{i=1}^{g_t} \pi_i 1 \left( Q_t^i = i, B_t^{g_t} \geq c_i, X_{r_i}^t = 1 \right) \right]
\]
\[
= \sum_{t=1}^{T} E \left[ \pi_i \sum_{r=1}^{g_t} 1 \left( Q_t^i = i, B_t^{g_t} \geq c_i, X_{r_i}^t = 1, S^t \geq r \right) \right] \]
\[
= \sum_{t=1}^{T} E \left[ \pi_i \sum_{r=1}^{g_t} 1 \left( Q_t^i = i, B_t^{g_t} \geq c_i, X_{r_i}^t = 1, S^t \geq r \right) \right] g_t \]
\[
= \sum_{t=1}^{T} E \left[ \sum_{i=1}^{g_t} \pi_i \sum_{r=1}^{g_t} 1 \left( B_t^{g_t} \geq c_i, S^t \geq r \right) \right] \]
\[
= \sum_{t=1}^{T} \sum_{i=1}^{g_t} \pi_i \lambda_p \sum_{r=1}^{g_t} 1 \left( B_t^{g_t} \geq c_i, S^t \geq r \right) \]

where the penultimate equality follows from Lemma 1.

Recall that \( F_t \in \{0, 1\} \) denotes a binary random variable indicating whether or not we choose a decision based on the prefix ordering in period \( t \), where \( F_t = 1 \) implies that \( g_t \) equals to \( \ell_t \) (the decision based on prefix ordering), and when \( F_t = 0 \), \( g_t \) is a random decision. Note that, by definition, \( F_t = 1 \) with probability \( 1 - \gamma_t \). Then, we have that

\[
\sum_{t=1}^{T} E[Z_{G_t}] = \sum_{t=1}^{T} \sum_{i=1}^{g_t} \pi_i \lambda_p \sum_{r=1}^{g_t} 1 \left( B_t^{g_t} \geq c_i, S^t \geq r \right) \]
\[
\geq \sum_{t=1}^{T} \sum_{i=1}^{g_t} \pi_i \lambda_p \sum_{r=1}^{g_t} 1 \left( B_t^{g_t} \geq c_i, S^t \geq r \right) \]
\[
= \sum_{t=1}^{T} \sum_{i=1}^{g_t} \pi_i \lambda_p \sum_{r=1}^{g_t} 1 \left( B_t^{g_t} \geq c_i, S^t \geq r \right) \]
\[
\geq \rho \left( 1 - \frac{1}{k(1+2s)/3} \right) \sum_{t=1}^{T} \sum_{i=1}^{g_t} \pi_i \lambda_p \sum_{r=1}^{g_t} 1 \left( B_t^{g_t} \geq c_i, S^t \geq r \right) \]

where the final inequality follows from the fact that when \( F_t = 1 \) and \( \ell_t \leq \ell_t \), we have that \( g_t = \ell_t \), which implies that

\[
E \left[ \sum_{i=1}^{g_t} \pi_i \lambda_p \sum_{r=1}^{g_t} 1 \left( B_t^{g_t} \geq c_i, S^t \geq r \right) \right] \]
\[
= \sum_{i=1}^{g_t} \pi_i \lambda_p E \left[ \sum_{r=1}^{g_t} 1 \left( B_t^{g_t} \geq c_i, S^t \geq r \right) \right] \]
\[
= \sum_{i=1}^{g_t} \pi_i \lambda_p E \left[ \sum_{r=1}^{g_t} 1 \left( B_t^{g_t} \geq c_i, S^t \geq r \right) \right] \]
\[
= \rho \left( 1 - \frac{1}{k(1+2s)/3} \right) \sum_{i=1}^{g_t} \pi_i \lambda_p \sum_{r=1}^{g_t} 1 \left( B_t^{g_t} \geq c_i, S^t \geq r \right) \]

where the inequality follows from Lemma 5 and the fact that, conditioned on \( \ell_t \) and \( S^t \geq r \), \( B_t^{g_t} \) is independent of \( F_t \). The penultimate equality follows from the fact that \( S^t \) is independent of \( \ell_t \) and \( F_t \). The final equality follows from the observation
Therefore, we have that
\[
\sum_{r=1}^{[\mu]} P \{ S^t \geq r \} = E \left[ \min \{ S^t, \mu \} \right] = \rho \mu.
\]

The above result implies that
\[
1 \{ F_t = 1 ; \; \ell_t \leq I_U \} \cdot E \left[ \sum_{r=1}^{g_t} \sum_{i=1}^{g_t} \pi_i \lambda_i p_i 1 \{ B_r^i \geq c_t \} \right] \ell_t, F_i \geq \rho \left( 1 - \frac{1}{k^{(1+2\alpha)/3}} \right) 1 \{ F_t = 1 ; \; \ell_t \leq I_U \} \mu \sum_{i=1}^{g_t} \pi_i \lambda_i p_i
\]
which gives the desired result.

Thus, we have that
\[
\sum_{t=1}^{T} E [\pi_{G_t}] \geq \rho \left( 1 - \frac{1}{k^{(1+2\alpha)/3}} \right) \sum_{t=1}^{T} E \left[ 1 \{ F_t = 1 ; \; \ell_t \leq I_U \} \cdot \mu \sum_{i=1}^{g_t} \pi_i \lambda_i p_i \right]
\]
\[
\geq \rho \left( 1 - \frac{1}{k^{(1+2\alpha)/3}} \right) Z^* \sum_{t=1}^{T} E \left[ 1 \{ F_t = 1 ; \; \ell_t \leq I_U \} \cdot \mu \sum_{i=1}^{g_t} c_i \lambda_i p_i \right]
\]
where the final inequality follows from the fact that if \( F_t = 1 \) and \( \ell_t \leq I_U \), then we have that with probability 1, \( g_t = \ell_t \) and by definition of \( I_U \),
\[
\mu \sum_{i=1}^{g_t} c_i \lambda_i p_i = \mu \sum_{i=1}^{\ell_t} c_i \lambda_i p_i \leq 1 - \frac{1}{k} - \frac{1}{k^{(1-\alpha)/3}}.
\]
and it follows from Lemma 4 that
\[
\sum_{i=1}^{\ell_t} \pi_i \lambda_i p_i \geq Z^* \sum_{i=1}^{\ell_t} c_i \lambda_i p_i.
\]
Therefore, we have that
\[
1 \{ F_t = 1 ; \; \ell_t \leq I_U \} \cdot \mu \sum_{i=1}^{g_t} \pi_i \lambda_i p_i \geq Z^* 1 \{ F_t = 1 ; \; \ell_t \leq I_U \} \cdot \mu \sum_{i=1}^{g_t} c_i \lambda_i p_i.
\]
Let \( C^* = 1 - 1/k - 2/k^{1-\alpha/3} - 1/k^{1-\alpha} \). Note that by definition of \( \ell_t \), we have that
\[
\mu \sum_{i=1}^{\ell_t} c_i \lambda_i \hat{p}_{i-1}^{t-1} \leq 1 - \frac{1}{k} - \frac{2}{k^{1-\alpha/3}} \leq \mu \sum_{i=1}^{\ell_t+1} c_i \lambda_i \hat{p}_{i-1}^{t-1},
\]
which implies that \( \mu \sum_{i=1}^{\ell_t} c_i \lambda_i \hat{p}_{i-1}^{t-1} \geq C^* \) since it follows from Assumption 1(c) that \( c_i \lambda_i \mu \leq 1/k^{1-\alpha} \) for all \( i \). Then, we have the following inequalities
\[
\sum_{t=1}^{T} E [Z_{G_t}] \geq \rho \left( 1 - \frac{1}{k^{(1+2\alpha)/3}} \right) Z^* \sum_{t=1}^{T} E \left[ 1 \{ F_t = 1 ; \; \ell_t \leq I_U \} \cdot \mu \sum_{i=1}^{g_t} c_i \lambda_i p_i \right]
\]
\[
= \rho \left( 1 - \frac{1}{k^{(1+2\alpha)/3}} \right) Z^* \sum_{t=1}^{T} E \left[ 1 \{ F_t = 1 ; \; \ell_t \leq I_U \} \left\{ \mu \sum_{i=1}^{g_t} c_i \lambda_i \hat{p}_{i-1}^{t-1} + \mu \sum_{i=1}^{g_t} c_i \lambda_i (p_i - \hat{p}_{i-1}^{t-1}) \right\} \right]
\]
\[
\geq \rho \left( 1 - \frac{1}{k^{(1+2\alpha)/3}} \right) Z^* \sum_{t=1}^{T} E \left[ 1 \{ F_t = 1 ; \; \ell_t \leq I_U \} \left\{ C^* + \mu \sum_{i=1}^{g_t} c_i \lambda_i (p_i - \hat{p}_{i-1}^{t-1}) \right\} \right]
\]
where the first equality follows from the fact that when $F_i = 1$, $g_i = \ell_i$. Therefore,

$$\sum_{t=1}^T E[Z_{G_i}] \geq \rho \left( 1 - \frac{1}{k^{(1+2\alpha)/3}} \right) Z^* C^* \sum_{t=1}^T E[1 \{F_i = 1 ; \ell_i \leq I_t\}]$$

$$+ \rho \left( 1 - \frac{1}{k^{(1+2\alpha)/3}} \right) Z^* E \left[ \sum_{t=1}^T 1 \{F_i = 1 ; \ell_i \leq I_t\} \sum_{i=1}^g c_i \lambda_i \mu (p_i - \hat{p}_i^{(t-1)}) \right]$$

$$\geq \rho \left( 1 - \frac{1}{k^{(1+2\alpha)/3}} \right) Z^* C^* \sum_{t=1}^T E[1 \{F_i = 1 ; \ell_i \leq I_t\}]$$

$$- \rho \left( 1 - \frac{1}{k^{(1+2\alpha)/3}} \right) Z^* E \left[ \sum_{t=1}^T 1 \{F_i = 1 ; \ell_i \leq I_t\} \sum_{i=1}^g c_i \lambda_i \mu |p_i - \hat{p}_i^{(t-1)}| \right]$$

$$= \rho \left( 1 - \frac{1}{k^{(1+2\alpha)/3}} \right) Z^* C^* \sum_{t=1}^T P\{F_i = 1\} \sum_{i=1}^g c_i \lambda_i \mu \gamma_i$$

$$- \rho \left( 1 - \frac{1}{k^{(1+2\alpha)/3}} \right) Z^* E \left[ \sum_{i=1}^g \sum_{t=1}^T 1 \{F_i = 1 ; \ell_i \leq I_t\} \sum_{t \leq T \gamma_i \{F_i = 1 ; \ell_i \leq I_t\}, t \geq 1} c_i \lambda_i \mu |p_i - \hat{p}_i^{(t-1)}| \right]$$

where the first equality follows from the fact that when $F_i = 1$, $g_i = \ell_i$. The second inequality follows from Lemma 8. Note that when $F_i = 1$, $g_i = \ell_i$. Using the fact that $c_i \lambda_i \mu \leq 1/k^{1-\alpha}$ for all $i$, we have that

$$E \left[ \sum_{i=1}^g \sum_{t \leq T \gamma_i \{F_i = 1 ; \ell_i \leq I_t\}, t \geq 1} c_i \lambda_i \mu \gamma_i \sum_{i=1}^g \sum_{t \leq T \gamma_i \{F_i = 1 ; \ell_i \leq I_t\}, t \geq 1} |p_i - \hat{p}_i^{(t-1)}| \right] \leq \mu \sum_{i=1}^g \sum_{t \leq T \gamma_i \{F_i = 1 ; \ell_i \leq I_t\}, t \geq 1} |p_i - \hat{p}_i^{(t-1)}|$$

Putting everything together, we have that

$$\frac{\sum_{t=1}^T E[Z_{G_i}]}{T Z^*} \geq \rho C^* \left( 1 - \frac{1}{k^{(1+2\alpha)/3}} \right) \left( 1 - \frac{1}{k^{(1-\alpha)/3}} \right) \frac{1}{T} \sum_{t=1}^T (1 - \gamma_i)$$

$$- \rho \mu \sum_{i=1}^g c_i \lambda_i \mu \sum_{t \leq T \gamma_i \{F_i = 1 ; \ell_i \leq I_t\}, t \geq 1} |p_i - \hat{p}_i^{(t-1)}|$$

$$\geq \rho \left( 1 - \frac{1}{k^{(1+2\alpha)/3}} - \frac{3}{k^{(1+2\alpha)/3}} \right) \frac{1}{T} \sum_{t=1}^T (1 - \gamma_i)$$

$$- \rho \mu \sum_{i=1}^g c_i \lambda_i \mu \sum_{t \leq T \gamma_i \{F_i = 1 ; \ell_i \leq I_t\}, t \geq 1} |p_i - \hat{p}_i^{(t-1)}|$$

$$\geq \rho \left( 1 - \frac{1}{k^{(1+2\alpha)/3}} - \frac{3}{k^{(1+2\alpha)/3}} \right) - \sum_{i=1}^g \frac{1}{T} \gamma_i$$

$$- \rho \mu \sum_{i=1}^g c_i \lambda_i \mu \sum_{t \leq T \gamma_i \{F_i = 1 ; \ell_i \leq I_t\}, t \geq 1} |p_i - \hat{p}_i^{(t-1)}|$$
where is the desired result. Note that the second inequality follows from the fact that

\[
C^* \left( 1 - \frac{1}{k(1+2n)^3} \right) \left( 1 - \frac{1}{k(1-n)^3} \right) \\
= \left( 1 - \frac{1}{k} - \frac{2}{k(1-n)^3} - \frac{1}{k(1+2n)^3} \right) \left( 1 - \frac{1}{k(1-n)^3} \right) \\
= \left( 1 - \frac{1}{k} - \frac{2}{k(1-n)^3} - \frac{1}{k(1+2n)^3} \right) \left( 1 - \frac{1}{k(1-n)^3} \right) + \frac{1}{k(1+2n)^3} + \frac{1}{k(1-n)^3} - \frac{1}{k(5+2n)^3} \\
+ \left( \frac{2}{k(1-n)^3} + \frac{2}{k(2+2n)^3} \right) \left( \frac{2}{k(2-2n)^3} - \frac{2}{k} \right) + \left( -\frac{1}{k} + \frac{1}{k(4-n)^3} + \frac{1}{k(4-4n)^3} - \frac{1}{k(5-2n)^3} \right) \\
= \left( 1 - \frac{1}{k(1+2n)^3} - \frac{3}{k(1-n)^3} + \frac{3}{k(2+2n)^3} - \frac{3}{k} \right) + \left( \frac{1}{k(4-n)^3} - \frac{1}{k(5+2n)^3} \right) \\
+ \left( \frac{2}{k(4-n)^3} - \frac{1}{k(2-2n)^3} \right) + \left( \frac{2}{k(2-2n)^3} - \frac{1}{k(5+2n)^3} \right) + \frac{1}{k(4-4n)^3} \\
\geq 1 - \frac{1}{k(1+2n)^3} - \frac{3}{k(1-n)^3}
\]

\[\Box\]

G.3 Proof of Lemma 10

The proof of Lemma 10 makes use of the following results. The first result provides an upper bound on the sample average for a Bernoulli random variable. Since the result follows from a standard application of Chernoff bound (cf. [20]), we omit the proof.

**Lemma 11.** Let \(X_1, X_2, \ldots, X_n\) be independent and identically distributed Bernoulli random variables with a parameter \(p\), i.e. for all \(i\), \(\mathcal{P}\{X_i = 1\} = 1 - \mathcal{P}\{X_i = 0\} = p\). Then, for any \(0 < \epsilon < 1\),

\[
E \left[ p - \frac{1}{n} \sum_{i=1}^{n} X_i \right] \leq \epsilon + 2e^{-\alpha^2/3}.
\]

The next result is a restatement of the Bernstein’s Inequality for real-valued random variables. For a proof, the reader is refer to Appendix B in [22].

**Theorem 3.** Let \(Y_1, \ldots, Y_n\) be independent nonnegative random variables and \(0 \leq Y_i \leq M\) for all \(i\). For any \(i\), let \(\sigma_i^2 = \text{Var}(Y_i)\). Let \(Y = \sum_{i=1}^{n} Y_i\) and \(\mu = E[Y]\). If \(V \geq \sum_{i=1}^{n} \sigma_i^2\), then for any \(0 < \delta < 1\),

\[
\mathcal{P}\{Y \leq (1 - \delta) \mu\} \leq \exp \left\{ -\frac{\delta^2 \mu^2}{V + \frac{1}{2} M \delta \mu} \right\}
\]

Using the fact that, if \(a, b\) and \(c\) are nonnegative, the function \(x \mapsto (ax^2)/(b + cx)\) is increasing in \(x\) for \(x \geq 0\), the corollary below follows immediately from the theorem.

**Corollary 1.** Let \(Y_1, \ldots, Y_n\) be independent nonnegative random variables and \(0 \leq Y_i \leq M\) for all \(i\). For any \(i\), let \(\sigma_i^2 = \text{Var}(Y_i)\). Let \(Y = \sum_{i=1}^{n} Y_i\) and \(\mu = E[Y]\). If \(V \geq \sum_{i=1}^{n} \sigma_i^2\), then for any \(0 < \delta < 1\) and \(0 \leq \eta \leq \mu\),

\[
\mathcal{P}\{Y \leq (1 - \delta) \eta\} \leq \exp \left\{ -\frac{\delta^2 \eta^2}{V + \frac{1}{2} M \delta \eta} \right\}
\]

The next lemma establishes an upper bound on the expected difference between \(p_i\) and \(\tilde{p}_i^{-1}\).

**Lemma 12.** For any keyword \(i\) and \(0 < \epsilon, \delta < 1\),

\[
E \left[ \sum_{t \leq T, \Omega_1 < 1(T_1 = 1, t_1 \leq T_2) \geq t} \left| p_i - \tilde{p}_i^{-1} \right| \right] \leq \epsilon T + \frac{1}{1 - \exp \left\{ -\delta^2 \lambda_i \mu^2 \left( 1 - \frac{1}{k(1+2n)^3} \right)^2 / 4 \right\}} \left( 1 - \exp \left\{ -\epsilon^2 (1 - \delta) \lambda_i \mu \left( 1 - \frac{1}{k(1+2n)^3} \right) / 3 \right\} \right).
\]
The random variable \( g_{t_h} \cdot 1 \{ F_{t_h} = 1, \ell_{t_h} \leq T_U \} \geq i \), \( h = 1, \ldots, w \).

Note that during time periods \( t_1, \ldots, t_w \), we have selected keywords based on the prefix ordering (i.e. \( F_{t_h} = 1 \) and \( g_{t_h} = \ell_{t_h} \) for all \( h \)) whose largest index \( g_{t_h} \) includes \( i \) and is less than or equal to \( T_U \).

We will show that, for any \( t_1, \ldots, t_w \) satisfying the above inequality, the conditional expectation

\[
E \left[ \sum_{t \leq T : g_t \cdot 1 \{ F_t = 1, \ell_t \leq T_U \} \geq i} |p_t - \hat{p}_i^{t-1}| \left| (t_1, \ell_{t_1}, F_{t_1}), \ldots, (t_w, \ell_{t_w}, F_{t_w}) \right| \right]
\]

is bounded above by the expression given in Lemma 12. For ease of exposition, we will drop the reference to the conditioning information \( (t_1, \ell_{t_1}, F_{t_1}), \ldots, (t_w, \ell_{t_w}, F_{t_w}) \) when it is clear from the context.

Recall that \( Y_i(t) \) denotes the total cumulative impressions that the ad associated with keyword \( i \) has received at the end of period \( t \). Conditioned on \( (t_1, \ell_{t_1}, F_{t_1}), \ldots, (t_w, \ell_{t_w}, F_{t_w}) \), we have that

\[
E \left[ \sum_{t \leq T : g_t \cdot 1 \{ F_t = 1, \ell_t \leq T_U \} \geq i} |p_t - \hat{p}_i^{t-1}| \left| (t_1, \ell_{t_1}, F_{t_1}), \ldots, (t_w, \ell_{t_w}, F_{t_w}) \right| \right] 
= \sum_{h=1}^{w} E \left[ \hat{p}_i^{t_h-1} - p_i \right] 
= \sum_{h=1}^{w} E \left[ \hat{p}_i^{t_h-1} - p_i \right] \cdot 1 \left( Y_i(t_h - 1) \leq (1 - \delta)(h - 1)\lambda_i \rho \mu \left( 1 - \frac{1}{k(1+2\alpha)^{3/4}} \right) \right) 
+ \sum_{h=1}^{w} E \left[ \hat{p}_i^{t_h-1} - p_i \right] \cdot 1 \left( Y_i(t_h - 1) > (1 - \delta)(h - 1)\lambda_i \rho \mu \left( 1 - \frac{1}{k(1+2\alpha)^{3/4}} \right) \right) 
\leq \sum_{h=1}^{w} P \left( Y_i(t_h - 1) \leq (1 - \delta)(h - 1)\lambda_i \rho \mu \left( 1 - \frac{1}{k(1+2\alpha)^{3/4}} \right) \right) 
+ \sum_{h=1}^{w} \left( \epsilon + 2 \exp \left\{-\epsilon^2(1 - \delta)(h - 1)\lambda_i \rho \mu \left( 1 - \frac{1}{k(1+2\alpha)^{3/4}} \right) / 3 \right\} \right) 
\leq \sum_{h=1}^{w} P \left( Y_i(t_h - 1) \leq (1 - \delta)(h - 1)\lambda_i \rho \mu \left( 1 - \frac{1}{k(1+2\alpha)^{3/4}} \right) \right) 
+ \epsilon T + \frac{2}{1 - \exp \left\{-\epsilon^2(1 - \delta)\lambda_i \rho \mu \left( 1 - \frac{1}{k(1+2\alpha)^{3/4}} \right) / 3 \right\}}
\]

(2)

where the first inequality follows from Lemma 11 and the final inequality follows from the formula for the geometric series.

Thus, to establish the desired result, it suffices to show that

\[
\sum_{h=1}^{w} P \left( Y_i(t_h - 1) \leq (1 - \delta)(h - 1)\lambda_i \rho \mu \left( 1 - \frac{1}{k(1+2\alpha)^{3/4}} \right) \right) \leq \frac{1}{1 - \exp \left\{-\epsilon^2(1 - \delta)\lambda_i \rho \mu \left( 1 - \frac{1}{k(1+2\alpha)^{3/4}} \right) / 4 \right\}}
\]

Recall that \( B_r^{G_t} \) denote the remaining account balance in period \( t_h \) just before the arrival of the \( r \)th search query (in period \( t_h \)), assuming that we bid on keywords in \( G_{t_h} = \{1, \ldots, g_{t_h} \} \) and \( Q_i^{t_h} \) denotes the keyword associated with the \( r \)th search query in period \( t_h \). For any \( 1 \leq h \leq w \), let \( U_h \) be defined by

\[
U_h = \min \left\{ S_i^{h} \mid [\mu] \right\} 
= \sum_{r=1}^{\infty} 1 \left( Q_i^{t_h} = i, B_r^{G_t} \geq c_i \right) 
= \sum_{r=1}^{\infty} 1 \left( Q_i^{t_h} = i, B_r^{G_t} \geq c_i, \min \{ S_i^{t_h}, [\mu] \} \geq r \right) 
= \sum_{r=1}^{[\mu]} 1 \left( Q_i^{t_h} = i, B_r^{G_t} \geq c_i, S_i^{t_h} \geq r \right)
\]

The random variable \( U_h \) provides a lower bound on the number of impressions that keyword \( i \) receives in period \( t_h \). Thus,
for any $1 \leq h \leq w$,
\[
\sum_{v=1}^{h-1} U_v \leq Y_h (t_h - 1).
\]

Note that, conditioned on $(t_1, \ell_1, F_{t_1}), \ldots, (t_w, \ell_w, F_{t_w})$, the random variables $U_1, U_2, \ldots, U_w$ are independent because $S_i$'s are i.i.d. and the distribution of $U_h$ depends only on $G_{t_h}$ which is fixed given our conditioning information. We can thus apply the result of Corollary 1 to the random variable $\sum_{v=1}^{h-1} U_v$. To do so, we need to determine its expectation and variance. Conditioned on $t_1, \ldots, t_w$, we will show that for any $1 \leq h \leq w$,
\[
E \left[ \sum_{v=1}^{h-1} U_v \right] \geq (h-1) \lambda_i \mu \left( 1 - \frac{1}{k(1+2\alpha)/\pi} \right) \quad \text{and} \quad \sum_{v=1}^{h-1} \text{Var} [U_v] \leq (h-1) \lambda_i \mu^2.
\]

To prove this result, note that for any $1 \leq h \leq w$,
\[
E [U_h] = \sum_{r=1}^{\lfloor \mu \rfloor} \lambda_i P \left\{ B_r^{G_{t_h}} \geq c_i, S_r^{G_{t_h}} \geq r \right\} \geq \lambda_i \left( 1 - \frac{1}{k(1+2\alpha)/\pi} \right) \sum_{r=1}^{\lfloor \mu \rfloor} P \{ S_r^{G_{t_h}} \geq r \} = \lambda_i E \left[ \min \{ S_r^{G_{t_h}}, \mu \} \right] \left( 1 - \frac{1}{k(1+2\alpha)/\pi} \right)
\]
\[
= \lambda_i \mu \left( 1 - \frac{1}{k(1+2\alpha)/\pi} \right),
\]
where the inequality follows from Lemma 5 and the fact that $G_{t_h} = \{1, 2, \ldots, g_h\}$ and $g_h \leq I_U$ for all $1 \leq h \leq w$. Note that since $g_h \leq I_U$ for all $h$, it follows from the definition of $I_U$ that
\[
\rho \sum_{u=1}^{g_h} c_u \lambda_u p_u \leq 1 - \frac{1}{k(1-\alpha)/\pi},
\]
which is the required hypothesis for the application of Lemma 5. The above result implies that for any $1 \leq h \leq w$,
\[
E \left[ \sum_{v=1}^{h-1} U_v \right] \geq (h-1) \lambda_i \mu \left( 1 - \frac{1}{k(1+2\alpha)/\pi} \right),
\]

Moreover, conditioned on $(t_1, \ell_1, F_{t_1}), \ldots, (t_w, \ell_w, F_{t_w})$, for any $1 \leq h \leq w$,
\[
\text{Var} [U_h] \leq E \left[ U_h^2 \right] = E \left[ \min \{ S_r^{G_{t_h}}, \lfloor \mu \rfloor \} \right] \left( \sum_{r=1}^{\lfloor \mu \rfloor} 1 \left( Q_r^{G_{t_h}} = i, B_r^{G_{t_h}} \geq c_i \right) \right)^2 \leq E \left[ \left( \sum_{r=1}^{\lfloor \mu \rfloor} 1 \left( Q_r^{G_{t_h}} = i \right) \right)^2 \right] = E \left[ \sum_{r=1}^{\lfloor \mu \rfloor} 1 \left( Q_r^{G_{t_h}} = i \right) + \sum_{1 \leq q, r \leq \lfloor \mu \rfloor, q \neq r} 1 \left( Q_r^{G_{t_h}} = i, Q_q^{G_{t_h}} = i \right) \right] = \lambda_i [\mu] + \lambda_i^2 [\mu] (|\mu| - 1) \leq \lambda_i \mu^2.
\]
where the last equality follows from the fact that, for $r \neq q$, the arrival of the $r^{th}$ and $q^{th}$ queries are independent of each other. The final inequality follows from our assumption that $\mu > 1$. The above result implies that
\[
\sum_{v=1}^{h-1} \text{Var} [U_v] \leq (h-1) \lambda_i \mu^2.
\]
Note that $0 \leq U_h \leq \mu$ for all $h$ and $i$. Therefore, by applying Corollary 1, we can conclude that for any $1 \leq h \leq w$,
\[
\mathcal{P}\left\{ Y_i(t_h - 1) \leq (1 - \delta)(h - 1)\lambda_i\mu\rho \left(1 - \frac{1}{k^{1+2\alpha} \delta^3}\right) \right\} \leq \mathcal{P}\left\{ \sum_{i=1}^{h-1} U_i \leq (1 - \delta)(h - 1)\lambda_i\mu\rho \left(1 - \frac{1}{k^{1+2\alpha} \delta^3}\right) \right\}
\leq \exp\left\{ -\frac{1}{2} \left[ \delta(h - 1)\lambda_i\mu\rho \left(1 - \frac{1}{k^{1+2\alpha} \delta^3}\right) \right]^2 \right\}
\leq \frac{-\frac{1}{2} \delta^2(h - 1)^2\lambda_i^2\mu^2\rho^2 \left(1 - \frac{1}{k^{1+2\alpha} \delta^3}\right)^2}{1 + \frac{1}{4} \delta^4 \rho \left(1 - \frac{1}{k^{1+2\alpha} \delta^3}\right)}
\leq \exp\left\{ -\delta^2(h - 1)\lambda_i^2\rho^2 \left(1 - \frac{1}{k^{1+2\alpha} \delta^3}\right)^2 \right\}
\]
where the last inequality follows from the fact that $1 + \frac{1}{4} \delta \left(1 - \frac{1}{k^{1+2\alpha} \delta^3}\right) \leq 2$.
Hence,
\[
\sum_{h=1}^{w} \mathcal{P}\left\{ Y_i(t_h - 1) \leq (1 - \delta)(h - 1)\lambda_i\mu\rho \left(1 - \frac{1}{k^{1+2\alpha} \delta^3}\right) \right\} \leq \sum_{h=1}^{w} \exp\left\{ -\frac{\delta^2(h - 1)\lambda_i^2\rho^2 \left(1 - \frac{1}{k^{1+2\alpha} \delta^3}\right)^2}{4} \right\}
\leq \frac{1}{1 - \exp\left\{ -\delta^2\lambda_i^2\rho^2 \left(1 - \frac{1}{k^{1+2\alpha} \delta^3}\right)^2 / 4 \right\}}
\]
where the last inequality follows from the standard bound for the geometric series. Since $(t_1, \ell_1, F_{t_1}), \ldots, (t_w, \ell_w, F_{t_w})$ are arbitrary, the desired result follows. \hfill \Box

Finally, here is the proof of Lemma 10.

**Proof.** It follows from Lemma 12 that
\[
\sum_{i=1}^{\ell_U} c_i \lambda_i \mu E \left[ \sum_{t_1}^{T_T} \sum_{1 \leq t \leq \ell_U} \left| p_{i_{\ell_U}} - p_{i_{t-1}} \right| \right]
\leq \sum_{i=1}^{\ell_U} c_i \lambda_i \mu T + \sum_{i=1}^{\ell_U} \frac{c_i \lambda_i \mu}{1 - \exp\left\{ -\delta^2\lambda_i^2\rho^2 \left(1 - \frac{1}{k^{1+2\alpha} \delta^3}\right)^2 / 4 \right\}} + \sum_{i=1}^{\ell_U} \frac{2c_i \lambda_i \mu}{1 - \exp\left\{ -\delta^2\lambda_i^2\rho^2 \left(1 - \frac{1}{k^{1+2\alpha} \delta^3}\right)^2 / 3 \right\}}
\leq \sum_{i=1}^{\ell_U} c_i \lambda_i \mu T + \sum_{i=1}^{\ell_U} \frac{8c_i \lambda_i \mu}{\delta^2\lambda_i^2\rho^2 \left(1 - \frac{1}{k^{1+2\alpha} \delta^3}\right)^2} + \sum_{i=1}^{\ell_U} \frac{6c_i \lambda_i \mu}{\delta^2\lambda_i^2\rho^2 \left(1 - \frac{1}{k^{1+2\alpha} \delta^3}\right)^2 (1 - e^{-\lambda_i \rho})}
\]
where the last inequality follows from the fact that for any $0 \leq x \leq 1$ and any $d \geq 0$,
\[
\frac{1}{1 - e^{-x}} \leq \frac{2}{x} \text{ and } \frac{1}{1 - e^{-dx}} \leq \frac{1}{x (1 - e^{-d})},
\]
which implies that
\[
\frac{1}{1 - \exp\left\{ -\delta^2\lambda_i^2\rho^2 \left(1 - \frac{1}{k^{1+2\alpha} \delta^3}\right)^2 / 4 \right\}} \leq \frac{8}{\delta^2\lambda_i^2\rho^2 \left(1 - \frac{1}{k^{1+2\alpha} \delta^3}\right)^2}
\]
\[1 - \exp\left\{-c^2(1 - \delta)\lambda_i\mu\left(1 - \frac{1}{k^{(1+2\alpha)/3}}\right)/3\right\} \leq c^2(1 - \delta)\rho\left(1 - \frac{1}{k^{(1+2\alpha)/3}}\right)(1 - e^{-\lambda_i\mu})\]

Using our assumption that \(c_i \leq 1/k\) and \(\lambda_i\mu \leq k^\alpha\) for all \(i\), we can conclude that for any \(0 < \epsilon, \delta < 1\),

\[
\sum_{i=1}^{T_U} c_i\lambda_i\mu E\left[\sum_{t \leq T: g_t = 1, F_t = 1, \ell_t \leq I_U) \geq t} p_i - \hat{p}_i^{t-1}\right] \leq \frac{\epsilon T I_U k^{1-\alpha}}{k^{(1+2\alpha)/3}} + \frac{8\mu I_U}{\delta^2 k^2 (1 - k^{(1+2\alpha)/3})^2} + \frac{6 I_U^2 \sum_{i=1}^{T_U} \lambda_i\mu/ (1 - e^{-\lambda_i\mu})}{k^{(7-4\alpha)/3}(1 - \delta)\rho (1 - k^{(1+2\alpha)/3})},
\]

which is the desired result.

Set \(\epsilon = k^{(2-2\alpha)/3}/I_U\). By definition of \(I_U\), it is easy to verify that \(\epsilon < 1\). Then, we have that for any \(0 < \delta < 1\),

\[
\sum_{i=1}^{T_U} c_i\lambda_i\mu E\left[\sum_{t \leq T: g_t = 1, F_t = 1, \ell_t \leq I_U) \geq t} p_i - \hat{p}_i^{t-1}\right] \leq \frac{T}{k^{(1-\alpha)/2}} + \frac{8\mu I_U}{\delta^2 k^2 (1 - k^{(1+2\alpha)/3})^2} + \frac{6 I_U^2 \sum_{i=1}^{T_U} \lambda_i\mu/ (1 - e^{-\lambda_i\mu})}{k^{(7-4\alpha)/3}(1 - \delta)\rho (1 - k^{(1+2\alpha)/3})},
\]

which gives the desired result. \(\square\)