Diffraction around Circular Canyon in Elastic Wedge Space by Plane SH-Waves

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Abstract: The wave propagation behavior in an elastic wedge-shaped medium, with a circular canyon at its vertex, has been studied. In particular, an analytic closed-form solution has been derived and solved for the case of plane shear horizontal- (SH) or source-emitted SH-wave incidence. The analysis demonstrates that the resulting surface displacement profile depends, as expected and as in similar analyses, on numerous parameters including the angle of the wedge, the frequency of the incident wave, the material properties of the media, and the angle of incidence. The results, especially in the context of varying wedge angles, provided intriguing results that help to explain geophysical observations regarding the amplification of seismic energy as a function of site conditions. The amplification resulting from a canyon in a sloping wedge space is now greater than two, exceeding that from the same canyon in a flat wedge space. The closed-form analytic solution also provides a set of results that can be used for comparison with those from approximation or other numerical methods.

INTRODUCTION

The research presented in this paper involves the analysis of plane SH-waves propagating in wedge-shaped media. Specifically, the geometry of the media studied ranges from a flat elastic half-space, where the wedge angle is 180°, through the sloping wedge space with wedge angles between 180° and 90° to a quarter-space in which the wedge angle equals 90°. Furthermore, it is assumed that a circular canyon exists at the vertex of the wedge angle. Fig. 1 illustrates the geometry of the sloping wedge space for the case of incident plane SH-waves.

Although the treatment of the problem is somewhat mathematical, it is believed that the consideration of such a problem has several important practical ramifications. For example, many homes and other structures in Southern California have been built on ridges and cliffs overlooking valleys and the ocean. The topography of these ridges can be reasonably characterized in two dimensions as a wedge space. Furthermore, the excavation of the foundation for these structures can be modeled as a circular canyon at the vertex of the wedge, where the horizontal and sloping surfaces intersect (see Fig. 1).

Thus, despite the analytic nature of the study discussed herein, the results do provide some helpful insights concerning the potential effects of earthquakes on structures located at specific sites and on varying media.

The proposed closed-form solution regarding the displacement fields generated under different conditions will also lend themselves nicely to approximate or numerical methods.

It will be seen that incident SH-waves traveling through this particular geometry result in displacement fields that depend on the angle of the wedge space, the angle of incidence, and the frequency of the incident wave train. This investigation studies the effect of varying these geometric and frequency-dependent parameters on the resulting propagation field.

An additional variable introduced into the evaluation of displacement fields in elastic wedges with a circular canyon at the vertex was the nature of the incident SH-wave.

It is believed that the wide spectrum of input parameters to the problem that have been studied provides constructive insights concerning the complicated nature of related wave-propagation phenomena.

A number of analytical studies have previously been performed concerning the propagation of waves—including acoustic, electromagnetic, and elastic—in wedge-shaped media. Many of these provide a helpful background to the present study and will be outlined in the following section.

PREVIOUS WORK

The earliest contributors to the study of elastic wave propagation can be found in Love's (1920) book on theory of elasticity. With regard to the propagation of waves in elastic wedge-shaped media, a comprehensive review of various analytical solutions was provided by Knopoff (1952). This compendium included numerous case studies and some elegant solutions based on Green's functions techniques and transform methods (Hudson 1963; Sato 1962).

MacDonald, in a 1902 publication entitled *Electric Waves*, discussed the propagation of electromagnetic waves in the context of boundary-valued problem solutions of the wave equation. His studies resulted in expressions for the total displacement field in a wedge in terms of Bessel function series expansions.

Using MacDonald's formulations, Sanchez-Sesma (1985) investigated the diffraction of elastic SH-waves in wedges. Sanchez-Sesma's studies included both time and frequency analyses to evaluate the surface displacement fields along the wedge perimeter. The displacement field arising from an SH-pulse (source) was analyzed by Abo-Zena and King (1973). Their publication, "SH-Pulse in an Elastic Wedge," studied the resulting pulse field using time-domain techniques.

The derivation of displacement fields in half-spaces with canyons of varying geometry was proposed by numerous authors (Trifunac 1971, 1972, 1973; Wong and Trifunac 1974a, 1974b, 1975; Luco et al. 1975; Todorovska and Lee 1990).

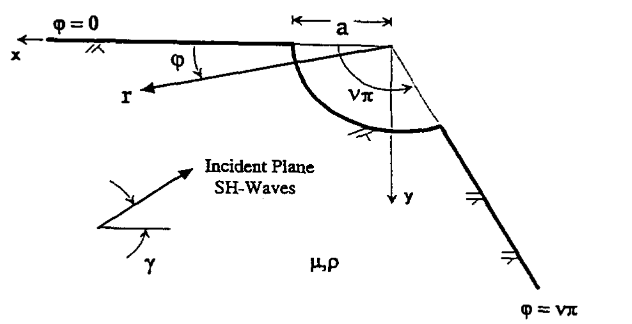

FIG. 1. Model

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Trifunac (1971) published his results concerning the "Surface Motion of a Semi-Cylindrical Alluvial Valley for Incident Plane SH-Waves." His study presents a closed-form solution for the displacement field and displays the complex propagation behavior encountered in this sort of problem. Trifunac (1973) also published "Scattering of Plane SH-Waves by a Semi-Cylindrical Canyon." This work presented a closed-form solution for the total displacement field for general angle of wave incidence and confirmed the importance of surface topography on the propagation of incident plane SH-waves.

WAVE PROPAGATION IN ELASTIC WEDGE

The previous section provided a qualitative description of numerous publications considered germane to the present investigation. In this section, the writers will devote special attention to those aforementioned studies upon which many of the analytical aspects of this work are based.

In describing the displacement field resulting from SH-wave propagation, it is generally necessary to consider both spatial and time-dependent parameters. For a shear wave speed β, the displacement field satisfies the wave equation given by

$$\beta^2\nabla^2 W = \frac{\partial^2 W}{\partial t^2} \quad (1)$$

for $W = w(x, y, t)$. However, if one assumes that the wave is harmonic with frequency ω such that

$$W = w(x, y)e^{-i\omega t} \quad (2)$$

then the wave equation degenerates to the Helmholtz wave equations

$$\nabla^2 w + (\omega/\beta)^2 w = 0 \quad (3)$$

Presently, it will be convenient to define the wave number k, which is given by

$$k = \omega/\beta \quad (4)$$

In the wedge-shape medium (Fig. 1), it is assumed that the material composing the medium is homogeneous and linearly elastic. The properties of the medium are defined by its rigidity μ and mass density ρ.

Note that both the rectangular and cylindrical coordinate systems with origin at the center of the circular canyon have been overlayed on the wedge geometry. The angle of the wedge is defined by $\varphi = \nu\pi$. Any point in the medium can be located using the polar (r, φ) coordinate. The angle of incidence of the SH-wave is given by the angle γ with respect to the horizontal direction. As noted earlier, the focus of this paper is to evaluate the displacement fields which result when plane SH-waves are incident to an elastic (wedge-shaped) sloping wedge space having a circular canyon at its vertex (Fig. 1). For each case studied it will be seen that this total displacement field (w) is the superposition of the individual displacement field for individual wave types arising in each case study. For the present case, the total displacement (w) is the sum of the incident SH-wave expression (w^i), the plane reflected wave (w^r), and the displacement field of the wave scattered off of and diffracted from the circular boundary (w^s).

Depending on the nature of the incident wave, it will be shown that the component of the total input displacement field that is common to all cases studied herein is the free-field displacements. Furthermore, the formulation for the free-field displacement does not depend on the presence of a circular canyon or valley at the vertex of the wedge. Therefore this free-field component can be considered as the displacement field that would prevail in the elastic wedge if topographic anomalies, such as circular canyons, did not exist. It should be evident, therefore, that the expression for the free-field component is critical to this investigation.

Of the publications reviewed in the previous section, the Sanchez-Sesma (1985) paper, outlining the mathematical formulation of the displacement field developed when plane SH-waves propagate in elastic wedges, is particularly helpful. A more thorough description of the formulations encountered through the following sections may be gleaned by first considering his results.

Sanchez-Sesma quantified the surface displacement field developed when SH-waves are propagated in an elastic wedge in the configuration shown in Fig. 1. Note that, unlike the geometry studied in the present investigation, Sanchez-Sesma's study does not include the circular canyon or valley at the vertex. Thus, by definition, his results do provide the expression for the free-field displacements, the input to the boundary-valued problem studied here.

Using MacDonald's (1902) results, Sanchez-Sesma provided the expression for the displacements (w) generated for plane SH-waves for plane SH-waves incident on an elastic semi-infinite medium with wedge angle ν. The resulting formulation is

$$\frac{w}{w_0} = \frac{2}{\nu}\sum_{n=0}^{\infty} \varepsilon_n e^{(-in\pi)/(2\nu)} J_{n/\nu}(kr)\cos\frac{n\gamma}{\nu}\cos\left(\frac{n\varphi}{\nu}\right) \quad (5)$$

for an input plane wave of amplitude w_0 with angle of incidence γ with respect to the horizontal. $J_{n/\nu}$ is the Bessel function of the first kind of order n/ν.

MODEL OF PLANE SH-WAVE INCIDENCE

The problem at hand can best be described through the illustration of Fig. 1 showing a train of plane SH-waves incident to a wedge-shaped half-space containing a circular canyon of radius a at its vertex. It is assumed that the material composing the half-space is homogeneous and linearly elastic. Incident waves will be reflected along the flat boundaries of the half-space and diffracted (scattered) from the circular canyon. The purpose of the immediate analysis will be to quantify the surface displacements along the half-space for various wedge angles and frequencies of incident waves.

As will be detailed shortly, it is simplest to attack the formulation by considering the separate displacement fields that arise in this problem. By superimposing these separate fields after applying the appropriate boundary conditions, the total displacement field at the surface of the wedge-shaped half-space can be evaluated.

Boundary Conditions

The boundary conditions for the present problem arise from the fact that waves diffract only in the soil medium. This implies that outside of the wedge-shaped half-space no seismic-wave propagation exists. Mathematically, the stress-free condition that exists along the entire surface of the half-space can be delineated between the circular surface along the canyon and the flat surfaces comprising the remainder of the wedge-shaped half space. Along the canyon surface one may express the stress-free boundary condition as

$$\tau_{rz} = \mu\left(\frac{\partial w}{\partial r}\right) = 0 \text{ at } r = a \quad (6)$$

Similarly, the flat surface outside the canyon is governed by the boundary condition

$$\tau_{z\varphi} = \frac{\mu}{r}\frac{\partial w}{\partial \varphi} = 0 \text{ at } \varphi = 0, \nu\pi \quad (7)$$

with $\varphi = 0$ corresponding to the flat surface of the wedge space and $\varphi = \nu\pi$ its sloping surface.

These boundary conditions will be applied to all the general expressions, each containing numerous undetermined coefficients, for the displacement field existing in the wedge space.

SCATTERED WAVE FIELD

Although the SH-waves incident to the circular canyon are plane in nature, the waves diffracted from the canyon are not plane and are classified as scattered waves. A requirement of all wave forms, whether plane or scattered in character, is that, being harmonic in nature, they satisfy the same Helmholtz wave equation [(3)]

$$\nabla^2 w(r, \varphi) + k^2 w(r, \varphi) = 0 \tag{8}$$

where

$$k = \omega/\beta = \omega/\sqrt{\mu/\rho} \tag{9}$$

is the wave number of the SH-waves.

Because of the geometric nature of the present problem, one can rewrite (8) using cylindrical coordinates such that

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2} + k^2 w = 0 \tag{10}$$

Notice that (10) is a second-order partial differential equation of in variables r and ϕ. A standard solution technique assumes that the solution representing the scattered wave form is separable with form

$$w^s = R(r)\Phi(\varphi)e^{-i\omega t} \tag{11}$$

The $e^{-i\omega t}$ term will be understood and omitted in every expression henceforth. Upon substituting this assumed solution into (10), one obtains two ordinary differential equations possessing solutions, for $n = 0, 1, 2, \ldots$

$$\Phi_n(\varphi) = A_n \cos\left(\frac{n\varphi}{\nu}\right) + B_n \sin\left(\frac{n\varphi}{\nu}\right) \tag{12}$$

$$R_n(r) = C_n H^{(1)}_{n/\nu}(kr) + D_n H^{(2)}_{n/\nu}(kr) \tag{13}$$

$H^{(1)}_{n/\nu}$ and $H^{(2)}_{n/\nu}$ shown in (13) are Hankel functions of the first and second kind of order n/ν. Notice that all functional orders and arguments have been transformed from the usual half-space (where $\nu = 1$) to the arbitrary wedge space (all ν) by substituting n/ν for all n. This is the same analytical modification that was made when extending the half-space formulation for the free-field excitation to the wedge-shaped space. In addition, the diffracted waves must radiate outwardly (away from the origin) from the canyon wall, satisfying the Sommerfield Radiation Condition. It is known that the $H^{(1)}_n(kr)e^{-i\omega t}$ function accommodates the radiation condition. The scattered wave field may now be described by

$$w^s(r, \varphi) = \sum_{n=0}^{\infty} [A_n \cos(n\varphi) + B_n \sin(n\varphi)][H^{(1)}_{n/\nu}(kr)] \tag{14}$$

Eq. (14) expresses the scattered wave field in terms of, as yet, undetermined coefficients. The coefficients will next be evaluated using the boundary conditions of the problem.

TOTAL DISPLACEMENT FIELD

The total displacement field is obtained by superimposing the free-field and scattered-wave expressions. Substituting the free-field displacements from (5), one obtains

$$w = w^f + w^s \tag{15}$$

$$w = \frac{2}{\nu}\sum_{n=0}^{\infty} e^{(-in\pi)/(2\nu)} J_{n/\nu}(kr)\cos\left(\frac{n\varphi}{\nu}\right)\cos\left(\frac{n\gamma}{\nu}\right)$$

$$+ \sum_{n=0}^{\infty}\left[A_n \cos\left(\frac{n\varphi}{\nu}\right) + B_n \sin\left(\frac{n\varphi}{\nu}\right)\right] H^{(1)}_{n/\nu}(kr) \tag{16}$$

the stress-free condition that exists along the flat and sloping surfaces of the space (where $\varphi = 0$ and $\nu\pi$), together with the orthogonality of the trigonometric conditions, reveals that the coefficient B_n must vanish and the total excitation reduces to

$$w = \frac{2}{\nu}\sum_{n=0}^{\infty} e^{(-in\pi)/(2\nu)} J_{n/\nu}(kr)\cos\left(\frac{n\varphi}{\nu}\right)\cos\left(\frac{n\gamma}{\nu}\right)$$

$$+ \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\varphi}{\nu}\right) H^{(1)}_{n/\nu}(kr) \tag{17}$$

Applying the stress-free condition (6) in effect along the surface of the circular canyon and using the orthogonality of $\cos(n\varphi/\nu)$ gives the final form of the total displacement field as

$$\frac{w}{w_0} = \frac{2}{\nu}\sum_{n=0}^{\infty} \varepsilon_n e^{-(in\pi)/(2\nu)}\left[J_{n/\nu}(kr) - H^{(1)}_{n/\nu}(kr)\frac{J'_{n/\nu}(ka)}{H^{(1)\prime}_{n/\nu}(ka)}\right]$$

$$\cdot\cos\left(\frac{n\varphi}{\nu}\right)\cos\left(\frac{n\gamma}{\nu}\right) \tag{18}$$

where $J'_{n/\nu}(\,)$ and $H'_{n/\nu}(\,)$ = derivatives of the corresponding Bessel and Hankel functions. In particular, for the case of a flat wedge space, where $\nu = 1$, (18) reduces to

$$\frac{w}{w_0} = 2\sum_{n=0}^{\infty} \varepsilon_n e^{(-in\pi)/2}\left[J_n(kr) - H^{(1)}_n(kr)\frac{J'_n(ka)}{H^{(1)\prime}_n(ka)}\right]$$

$$\cdot\cos\left(\frac{n\gamma}{\nu}\right)\cos\left(\frac{n\varphi}{\nu}\right) \tag{19}$$

Eq. (18) provides an explicit expression for the total displacement in the sloping wedge space when plane SH-waves are incident on the medium. Thus, at any point within the medium the displacement field can be evaluated.

The most useful results generated from the expression for total excitation within the wedge space are the displacements along the surface. This surface includes the flat and sloping portions where $\varphi = 0$ and $\varphi = \nu\pi$ and along the surface of the canyon $r = a$ (see Fig. 1).

The frequency of the incoming wave is specified by the dimensionless parameter

$$\eta = \frac{ka}{\pi} = \frac{\omega a}{\pi\beta} = \frac{2a}{\lambda} \tag{20}$$

where λ = wavelength of the incoming waves. Thus, the dimensionless frequency can also be considered as the ratio of the canyon diameter to the wavelength of the incoming waves.

Fig. 2 displays a three-dimensional plot of surface displacement amplitudes as a function of dimensionless distance x/a and dimensionless frequency η of the incident plane wave, where

$$\text{Displacement} = \sqrt{[\mathrm{Re}(w)]^2 + [\mathrm{Im}(w)]^2} \tag{21}$$

With respect to the rectangular coordinate system denoted in Fig. 1, it can be seen that x/a is a dimensionless parameter expressing the ratio of the distance from the origin (x) to the radius (a) of the canyon. Thus, $x/a = -1.0$ refers to the left edge of the canyon and $x/a = +1.0$ refers to the right side. The region $-1 \leq x/a \leq 1$ represents the circular region along the canyon perimeter.

The wedge space in the plot has a wedge angle of $\nu = 1$; in other words, it is a flat wedge space, the case studied earlier (Trifunac 1973). Since the same equations are used and the same solutions are obtained, the results are identical with that of Trifunac.

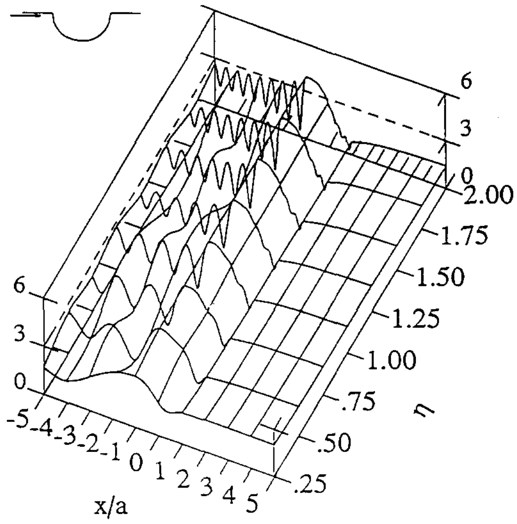

FIG. 2. Surface Displacements: Wedge Angle = 180°; Incidence Angle = 0°

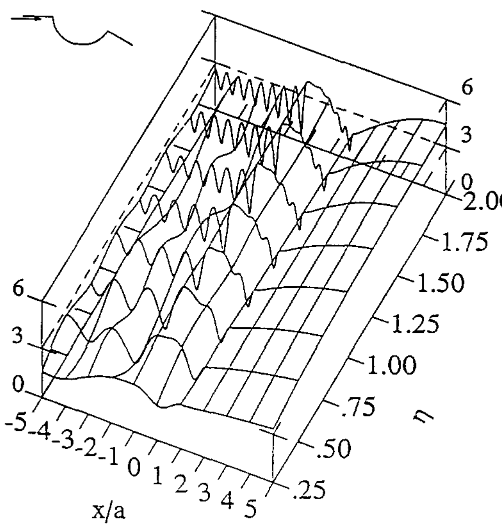

FIG. 3. Surface Displacements: Wedge Angle = 150°; Incidence Angle = 0°

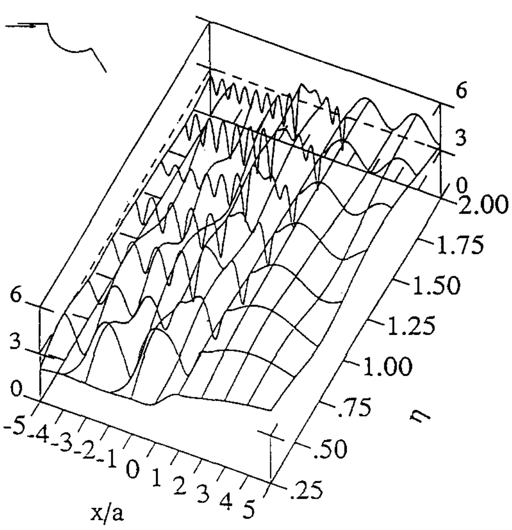

FIG. 4. Surface Displacements: Wedge Angle = 120°; Incidence Angle = 0°

The graphical results for the problem of plane SH-wave incidence, depicting the surface displacement amplitudes, reveal some interesting characteristics. The primary physical parameters which were modulated in an attempt to determine their affects on the nature of the wave propagation include the wedge angle ($\nu\pi$), the incident wave frequency (η), and the angle of incidence (γ). Numerous observations and conclusions can be set forth by considering each of these three variables separately.

Consider the propagation effects produced by varying the angle of the sloping wedge space (ν) while maintaining the frequency and incidence angle constant (see Figs. 2–5). It is evident that the surface displacements for $(x/a) > 1$ increase and begin to oscillate with periodic behavior as the wedge angle is decreased (as ν tends from 1 towards 1/2). This result is as expected because the effective area of "shadow zone," due to the presence of the circular canyon, is reduced as ν decreases. It should also be noted that the largest amplification

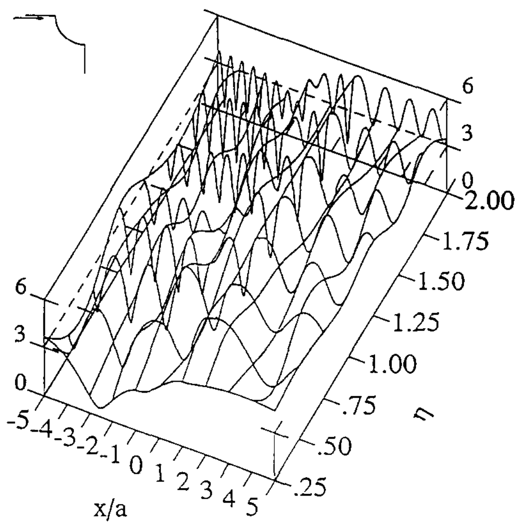

FIG. 5. Surface Displacements: Wedge Angle = 90°; Incidence Angle = 0°

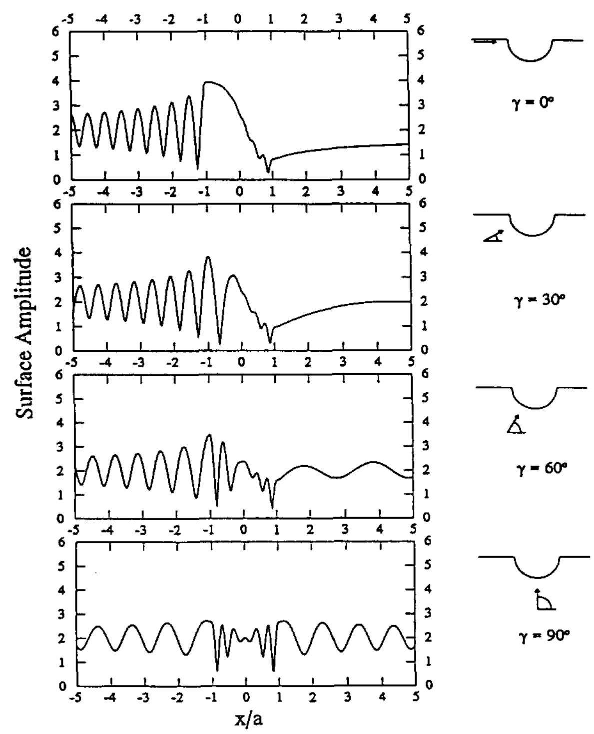

FIG. 6. Plane-Wave Incidence: 180° Wedge; η = 2.0

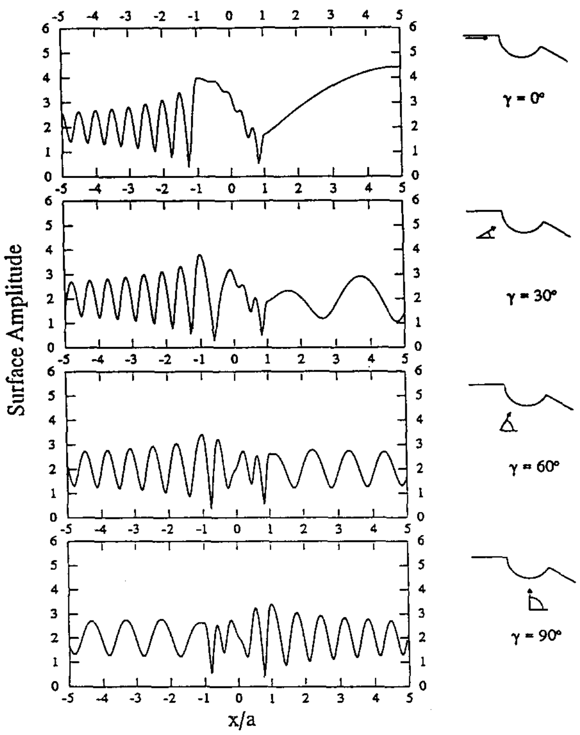

FIG. 7. Plane-Wave Incidence: 150° Wedge; η = 2.0

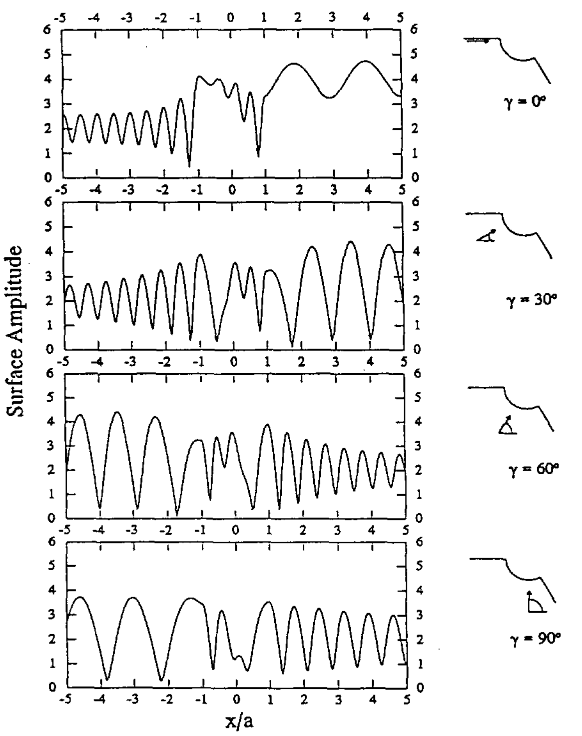

FIG. 8. Plane-Wave Incidence: 120° Wedge; η = 2.0

of the displacement amplitude is equal to $2/\nu$, which represents an amplification of $1/\nu$.

For a given wedge angle and frequency of incident wave train, conclusions regarding the effect of the angle of incidence can be drawn (see Figs. 6–9). First, the surface displacements are indeed symmetric with respect to $x/a = 0.0$ when the angle of incidence is half the wedge angle ($\gamma = \nu\pi/2$). Additionally, several cases of antisymmetry are available from the results that intuitively support the correctness of the computations. Such cases occur, for instance, when the wedge angle is 120°

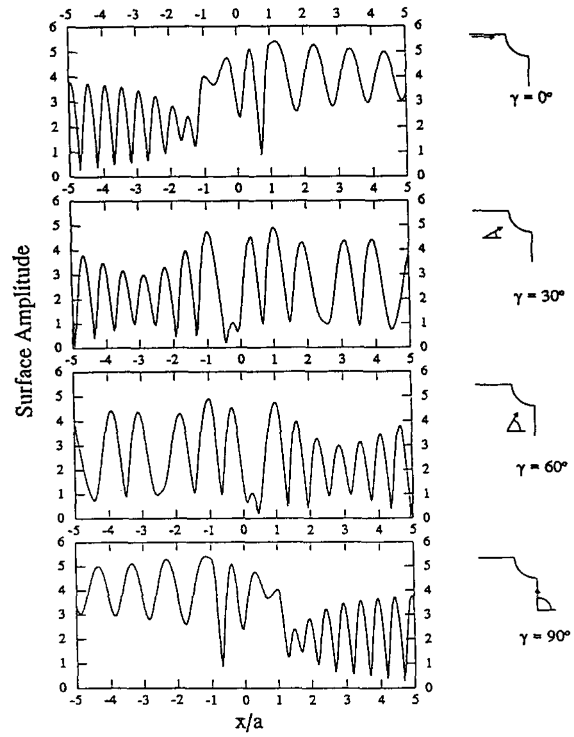

FIG. 9. Plane-Wave Incidence: 90° Wedge; η = 2.0

and incident angles are 30° and 90°. The surface displacements concur with the geometric antisymmetry of this case.

Finally, one can observe the effect of modulating the incoming waves' frequency content while fixing the wedge angle and incidence angle (refer to Figs. 3–6). The intuitive suggestion that the complexity of the total excitation increases as the frequency of the incident wave increases is confirmed from the analytical results. This augmented frequency of the total displacement field applies to the entire surface contour from $x/a = -5.0$ to $x/a = 5.0$.

It was noted that the solution for the problem of plane SH-waves incident to a half-space ($\nu = 1.0$) with a semicircular canyon at the surface has been previously solved.

The algorithms used presently provided identical results to the study done previously. That the present results agree identically with the previous study helps to confirm the latest results for wedge-shaped spaces at various angles.

CONCLUSIONS

The surface displacement fields generated by plane and source-emitted SH-waves in a wedge-shaped medium, with a circular canyon at its vertex, have been studied in this report. A closed-form solution has been derived to express these displacement fields at any point along the surfaces of the wedge space. The analysis has demonstrated that the surface displacement profiles depend on numerous parameters including the angle of the wedge, the frequency of the incident wave, the material properties of the medium, and the angle of incidence.

Important physical implications can be gleaned from the analytical results presented. First, exact solutions for the surface displacements of such a topography can be obtained. As noted, the wedge medium with circular canyon at its vertex does a reasonable job of characterizing the topography of many structures overlooking steep ridges and cliffs. Second, the exact solutions indeed demonstrate that the amplitude of incident wave can be amplified by as much as a factor of five. Third, the maximum amplifications tend to occur at the points where the circular canyon and flat surfaces intersect, where $x/$

$a = \pm 1.00$. Certainly, these results should be considered in the design of associated structures and foundations.

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