EIGENVALUE-EIGENVECTOR METHOD

1 Real Eigenvalues

EXAMPLE 1 Find a general solution of

$$x' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix} x. \tag{1.1}$$

The characteristic equations is

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -2 & 1 & 2-\lambda \end{vmatrix} = \lambda^2(2-\lambda) - 2 + \lambda = (2-\lambda)(\lambda^2 - 1).$$

Hence $\lambda_1 = -1,\ \lambda_2 = 1$, and $\lambda_3 = 2$ are eigenvalues. For $\lambda_1 = -1$ the eigenvector equation is $(A - (-1)I)v = 0$, that is,

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & 1 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 + v_2 \\ v_2 + v_3 \\ -2v_1 + v_2 + 3v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since $v_1 + v_2 = 0$, then $v_1 = -v_2$ and $v_2 + v_3 = 0$ gives $v_3 = -v_2$. The third equation is redundant. (This is the same as saying that the last row can be reduced to zeros by Gaussian elimination). However, as a check against algebra errors we verify that the third equation is true, that is,

$$-2v_1 + v_2 + 3v_3 = -2(-v_2) + v_2 + 3(-v_2) = 0.$$

Hence $v_1 = (-v_2, v_2, -v_2)^T$, where v_2 is any arbitrary *nonzero* constant. For example, we can choose $v_2 = -1$ so that

$$x_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{-t}.$$

For $\lambda_2 = 1$ the eigenvector equation is

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -v_1 + v_2 \\ -v_2 + v_3 \\ -2v_1 + v_2 + v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now $-v_1 + v_2 = 0$ gives $v_1 = v_2$ and $-v_2 = v_3 = 0$ gives $v_3 = v_2$ or $v = (v_2, v_2, v_2)^T$. The last equation is redundant but we check it, that is, $-2v_1 + v_2 + v_3 = -2v_2 + v_2 + v_2 = 0$. Since any *nonzero* v_2 will do, we use the convenient value $v_2 = 1$. This gives

$$x_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{t}.$$

For $\lambda_3 = 2$ the eigenvector equation is

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -2v_1 + v_2 \\ -2v_2 + v_3 \\ -2v_1 + v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence $v_2 = 2v_1$ and $v_3 = 2v_2 = 2(2v_1) = 4v_1$ or $v = (v_1, 2v_1, 4v_1)^T$, where v_1 is an *nonzero* constant. We take $v_1 = 1$ so that

$$x_3 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} e^{2t}.$$

A general solution is

$$\begin{aligned} x(t) &= c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} e^{2t} \\ &= \begin{pmatrix} e^{-t} & e^{t} & e^{2t} \\ -e^{-t} & e^{t} & 2e^{2t} \\ e^{-t} & e^{t} & 4e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \end{aligned} \tag{1.2}$$

There are infinitely many sets of fundamental solutions and hence infinitely many fundamental matrices and infinitely many general solutions. For example, another set of fundamental solutions is

$$\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} e^{-t}, \quad \begin{pmatrix} \frac{1}{2} \\ 1 \\ 2 \end{pmatrix} e^{2t}, \quad \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} e^{t}.$$

Hence a general solution is

$$x(t) = d_1 \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} e^{-1} + d_2 \begin{pmatrix} \frac{1}{2} \\ 1 \\ 2 \end{pmatrix} e^{2t} + d_3 \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} e^{t}.$$

2 Complex Eigenvalues

EXAMPLE 2 Find a real-valued set of fundamental solutions of

$$x' = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} x. \tag{2.1}$$

The characteristic equation is $\lambda^2 - 4\lambda + 13 = 0$. The characteristic roots are $\lambda_1 = 2 + 3i$ and $\lambda_2 = 2 - 3i$. For $\lambda_1 = 2 + 3i$ the eigenvector equation is

$$\begin{pmatrix} -3i & 3 \\ -3 & -3i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -3iv_1 + 3v_2 \\ -3v_1 - 3iv_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The solution is $v_2 = iv_1$, where v_1 is any nonzero constant. We take $v_1 = 1$ so that $v = (1, i)^T$ is an eigenvector. A complex-valued solution is

$$\begin{aligned}
x_1(t) &= e^{(2+3i)t}\begin{pmatrix} 1 \\ i \end{pmatrix} \\
&= e^{2t}(\cos 3t + i\sin 3t)\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] \\
&= e^{2t}\left[\cos 3t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\cos 3t\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\sin 3t\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin 3t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] \\
&= e^{2t}\left[\begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix} + i\begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}\right].
\end{aligned}$$

Two linearly independent real-valued solutions are

$$e^{2t}\begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix} \text{ and } e^{2t}\begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}.$$

The general solution is

$$x(t) = c_1 e^{2t}\begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}$$

or

$$x(t) = \begin{pmatrix} e^{2t}\cos 3t & e^{2t}\sin 3t \\ -e^{2t}\sin 3t & e^{2t}\cos 3t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = X(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

This fundamental matrix $X(t)$ satisfies $X(0) = I$. Hence

$$X(t) = e^{At} = \begin{pmatrix} e^{2t}\cos 3t & e^{2t}\sin 3t \\ -e^{2t}\sin 3t & e^{2t}\cos 3t \end{pmatrix}.$$

EXAMPLE 3 Find a real-valued fundamental set of solution of

$$x' = \begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} x.$$

The characteristic equation is

$$\begin{vmatrix} 1-\lambda & 2 & 0 \\ -1 & -1-\lambda & 0 \\ 1 & 0 & -1-\lambda \end{vmatrix} - (\lambda+1)(\lambda^2+1) = 0$$

Hence the eigenvalues are $\lambda_1 = i,\ \lambda_2 = -i$, and $\lambda_3 = -1$. For $\lambda_1 = i$ the eigenvector equation is

$$\begin{pmatrix} 1-i & 2 & 0 \\ -1 & -(1+i) & 0 \\ 1 & 0 & -(1+i) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} (1-i)v_1 + 2v_2 \\ -v_1 - (1+i)v_2 \\ v_1 - (1+i)v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The solution is $v_1 = -(1+i)v_2$ and $v_3 = (1+i)^{-1}v_1 = -v_2$. We choose $v_2 = -1$. Thus $v_1 = (1+i, -1, 1)^T$ is an eigenvector. Without further computation we can take $v_2 = \bar{v}_1 = (1-i, -1, 1)^T$ for $\lambda_2 = -i$. For $\lambda_3 = -1$, the eigenvector equation is

$$\begin{pmatrix} 2 & 2 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 2v_1 + 2v_2 \\ -v_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence $v_1 = 0,\ v_2 = 0$, and v_3 is arbitrary. We take $v_3 = 1$ so that $v_3 = (0,0,1)^T$. Since

$$\begin{aligned} \begin{pmatrix} 1+i \\ -1 \\ 1 \end{pmatrix} e^{it} &= \left[\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] (\cos t + i \sin t) \\ &= \left[\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cos t - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \sin t \right] + i \left[\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \sin t + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cos t \right] \\ &= \begin{pmatrix} \cos t - \sin t \\ -\cos t \\ \cos t \end{pmatrix} + i \begin{pmatrix} \sin t + \cos t \\ -\sin t \\ \sin t \end{pmatrix}, \end{aligned}$$

then a fundamental set of solution is

$$x_1(t) = \begin{pmatrix} \cos t - \sin t \\ -\cos t \\ \cos t \end{pmatrix}, \quad x_2(t) = \begin{pmatrix} \sin t + \cos t \\ -\sin t \\ \sin t \end{pmatrix}, \quad x_3(t) = \begin{pmatrix} 0 \\ 0 \\ e^{-t} \end{pmatrix}.$$

If A has more than one complex eigenvalue, the same technique can be applied separately to each eigenvalue.

EXAMPLE 4 Find a real-valued set of fundamental solutions of

$$x' = \begin{pmatrix} 0 & 1 & 1 & -3 \\ 0 & 3 & 0 & 1 \\ -1 & 3 & 0 & 1 \\ 0 & -1 & 0 & 3 \end{pmatrix} x. \tag{2.2}$$

The characteristic equation is

$$\begin{aligned} \begin{vmatrix} -\lambda & 1 & 1 & -3 \\ 0 & 3-\lambda & 0 & 1 \\ -1 & 3 & -\lambda & 1 \\ 0 & -1 & 0 & 3-\lambda \end{vmatrix} &= -\lambda \begin{vmatrix} 3-\lambda & 0 & 1 \\ 3 & -\lambda & 1 \\ -1 & 0 & 3-\lambda \end{vmatrix} \\ &+ (-1) \begin{vmatrix} 1 & 1 & -3 \\ 3-\lambda & 0 & 1 \\ -1 & 0 & 3-\lambda \end{vmatrix} \\ &= (-\lambda)[3-\lambda)^2(-\lambda) - \lambda] - [-1 - (3-\lambda)^2] \\ &= (\lambda^2+1)[(3-\lambda)^2 + 1] = (\lambda^2+1)(\lambda^2 - 6\lambda + 10). \end{aligned}$$

The eigenvalues are $\pm i$ and $3 \pm i$. For $\lambda = i$ the eigenvector equation is

$$\begin{pmatrix} -i & 1 & 1 & -3 \\ 0 & 3-i & 0 & 1 \\ -1 & 3 & -i & 1 \\ 0 & -1 & 0 & 3-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} -iv_1 + v_2 + v_3 - 3v_4 \\ (3-i)v_2 + v_4 \\ -v_1 + 3v_2 - iv_3 + v_4 \\ -v_2 + (3-i)v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

A solution is $v_2 = v_4 = 0,\ v_3 = iv_1$. Hence $v = (1, 0, i, 0)^T$ will do. A solution of (2.2) is

$$e^{it}\begin{pmatrix}1\\0\\i\\0\end{pmatrix} = (\cos t + i\sin t)\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + i\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right] = \begin{pmatrix}\cos t\\0\\-\sin t\\0\end{pmatrix} + i\begin{pmatrix}\sin t\\0\\\cos t\\0\end{pmatrix}. \tag{2.3}$$

For $\lambda = 3 + i$ the eigenvector equation is

$$\begin{pmatrix}-3-i & 1 & 1 & -3\\ 0 & -i & 0 & 1\\ -1 & 3 & -3-i & 1\\ 0 & -1 & 0 & -i\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\\v_4\end{pmatrix} = \begin{pmatrix}-(3+i)v_1 + v_2 + v_3 - 3v_4\\ -iv_2 + v_4\\ -v_1 + 3v_2 - (3+i)v_3 + v_4\\ -v_2 - iv_4\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}.$$

The solution is $v_4 = iv_2,\ v_1 = (9 - 7i)v_2/(9 + 6i),\ v_3 = (7 + 9i)v_2/(9 + 6i)$. An eigenvector is $v = (9 - 7i,\ 9 + 6i,\ 7 + 9i,\ -6 + 9i)^T$ and a solution of (2.2) is

$$\begin{aligned}&e^{3t}(\cos t + i\sin t)\left[\begin{pmatrix}9\\9\\7\\-6\end{pmatrix} + i\begin{pmatrix}-7\\6\\9\\9\end{pmatrix}\right]\\ &= e^{3t}\begin{pmatrix}9\cos t + 7\sin t\\ 9\cos t - 6\sin t\\ 7\cos t - 9\sin t\\ -6\cos t - 9\sin t\end{pmatrix} + ie^{3t}\begin{pmatrix}9\sin t - 7\cos t\\ 9\sin t + 6\cos t\\ 7\sin t + 9\cos t\\ -6\sin t + 9\cos t\end{pmatrix}.\end{aligned} \tag{2.4}$$

Given (2.3) and (2.4), we see that a real-valued fundamental set of solution is

$$\begin{pmatrix}\cos t\\0\\-\sin t\\0\end{pmatrix},\ \begin{pmatrix}\sin t\\0\\\cos t\\0\end{pmatrix},\ e^{3t}\begin{pmatrix}9\cos t + 7\sin t\\ 9\cos t - 6\sin t\\ 7\cos t - 9\sin t\\ -6\cos t - 9\sin t\end{pmatrix},\ e^{3t}\begin{pmatrix}9\sin t - 7\cos t\\ 9\sin t + 6\cos t\\ 7\sin t + 9\cos t\\ -6\sin t + 9\cos t\end{pmatrix}.$$