```
         ENDIF
         CBY(K)=CYY
 55   CONTINUE
      CDY(0)=-CBY(1)
      DO 60 K=1,NM                     <<<Calculate the derivative of Yn
 60      CDY(K)=CBY(K-1)-K/Z*CBY(K)
      RETURN
      END
```

5.5 COMPUTATION OF $J_\nu(z)$ AND $Y_\nu(z)$ WITH AN ARBITRARY ORDER

In this section we discuss the computation of the Bessel functions of an arbitrar order and present two computer programs: one for real arguments and the othe for complex arguments. The Bessel functions of an arbitrary order can still b computed using the recurrence relation (5.1.21) with the starting values eva uated from a series expansion for small arguments and an asymptotic expansio for large arguments. To be more specific, given an order ν, we can first fin ν_0 such that $\nu = \nu_0 + n$, where $0 \leq \nu_0 < 1$ and n is an integer. Then fc small arguments, we can evaluate $J_{\nu 0}(z)$, $J_{\nu 0+1}(z)$, $Y_{\nu 0}(z)$, and $Y_{\nu 0+1}(z)$ using th series expansions given by (5.1.2)–(5.1.4), which can be rewritten in a forr more suitable for numerical computation as

$$J_{\pm\nu}(z) = \frac{(z/2)^{\pm\nu}}{\Gamma(1 \pm \nu)} \left\{ 1 + \sum_{m=1}^{\infty} \prod_{k=1}^{m} \left[-\frac{(z/2)^2}{k(k \pm \nu)} \right] \right\} \qquad (5.5.1$$

For large arguments, we can evaluate $J_{\nu 0}(z)$, $J_{\nu 0+1}(z)$, $Y_{\nu 0}(z)$, and $Y_{\nu 0+1}(z)$ usin the asymptotic expansions given by (5.2.5) and (5.2.6), with (5.2.7) and (5.2.8 being evaluated directly since ν is now an arbitrary number. For the conve nience of numerical computation, (5.2.7) and (5.2.8) can be rewritten as

$$P(\nu, z) \sim 1 + \sum_{k=1}^{M_k} \prod_{j=1}^{k} \left\{ -\frac{[\mu - (4j-3)^2]\,[\mu - (4j-1)^2]}{2j(2j-1)\,(8z)^2} \right\} \qquad (5.5.2$$

$$Q(\nu, z) \sim \frac{\mu - 1}{8z} \left(1 + \sum_{k=1}^{M_k} \prod_{j=1}^{k} \left\{ -\frac{[\mu - (4j-1)^2]\,[\mu - (4j+1)^2]}{2j(2j+1)\,(8z)^2} \right\} \right)$$

(5.5.3

Once $J_{\nu 0}(z)$, $J_{\nu 0+1}(z)$, $Y_{\nu 0}(z)$, and $Y_{\nu 0+1}(z)$ are evaluated, we can compute $J_\nu(z$ by backward recurrence and $Y_\nu(z)$ by forward recurrence for a real z.

The following computer program implements the algorithm described abov and evaluates the Bessel functions of an arbitrary order with real arguments