Particular solution for systems – special case :

The following technique is generally shorter than the Variation of Parameters method. It is, however, limited to forcing terms that are sums of real or complex exponentials times constant vectors (see **Superposition** on the last page).
Consider the n-dimensional linear non-homogeneous (forced) system

$$\frac{dy}{dt} = Ay + e^{\sigma t}v \tag{1}$$

where $v \in \mathbb{R}^n$ and $\sigma = \alpha + i\beta$. We write this as

$$L[y] = e^{\sigma t}v \qquad \text{where} \qquad L[y] = \frac{dy}{dt} - Ay \tag{2}$$

Lemma 1 Let A be a real $n \times n$ matrix and let $y = \varphi + i\psi$ be a solution of

$$L[y] = g(t)$$

where $g = g_1(t) + ig_2(t)$. Then $L[\varphi] = g_1(t)$ and $L[\psi] = g_2(t)$.

Proof Use linearity. QED

Thus, for example, the solution of $L[y] = e^{\alpha t}\cos(\beta t)\ v$ can be obtained by solving $L[z] = e^{(\alpha+i\beta)t}\ v$ and then

$$y(t) = Re\ z(t).$$

In solving our problem we will treat two cases.

1. $\det(\mathbf{A} - \sigma\mathbf{I}) \neq \mathbf{0}$, ($\sigma$ not a root of the characteristic polynomial of A): We look for a solution of

   in the form $y = e^{\sigma t}\ u$ where $u \in \mathbb{R}^n$ is to be determined. Thus, again using linearity,

   $$L[y] = L[e^{\sigma t}\ u] = e^{\sigma t}(\sigma I - A)u \tag{3}$$

   Equating (2) and (3), we obtain

   $$(\sigma I - A)u = v \qquad \text{or} \qquad u = (\sigma I - A)^{-1}v.$$

2. $\det(\mathbf{A} - \sigma\mathbf{I}) = \mathbf{0}$ but σ is a **simple** root of the characteristic polynomial of A: Motivated by our treatment of homogeneous systems with defective matrices, we look for a solution of the form

   $$y(t) = e^{\sigma t}(c\,u\,t + w)$$

   where c is a constant, $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$, all to be determined. Computing,

   $$L[y] = L[e^{\sigma t}(c\,u\,t + w)] = e^{\sigma t}[c(\sigma I - A)u\,t + \sigma w + cu - Aw] \tag{4}$$

   Equating (2) and (4) and canceling the exponential, we obtain

   $$c(\sigma I - A)u\,t + \sigma w + cu - Aw = v$$

which must hold for all $t \in \mathbb{R}$. Thus

$$(\sigma I - A)u = 0, \qquad \text{and} \qquad (\sigma I - A)w = v - c\,u, \tag{5}$$

i.e. u must be an eigenvector corresponding to σ. Although the constant c can be determined algebraically (so that the second system is consistent), the technique uses concepts not covered in this course. Thus we let c be determined within each application as we now illustrate with

Problem 1 of Section 7.6
Find a particular solution of

$$\frac{dy}{dt} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} y + e^t \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$$

We see that $\sigma = 1$ and $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigen-pair and the other eigenvalue is $\lambda = 3$. Thus we are in Case No. 2 above and the second relation in (5) yields (after multiplying by a minus sign),

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} w = c \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$$

The augmented matrix is

$$\begin{bmatrix} 1 & -1 & c \\ -1 & 1 & c-4 \end{bmatrix}.$$

After adding row 1 to row 2 yields

$$\begin{bmatrix} 1 & -1 & c \\ 0 & 0 & 2c-4 \end{bmatrix}$$

from which it follows that $c = 2$ in order to have a consistent system. The remaining equation is

$$w_1 - w_2 = 2$$

and choosing $w_1 = 1$ and $w_2 = -1$ yields our final solution:

$$y_p(t) = e^{\sigma t}(c\,u\,t + w) = \begin{bmatrix} e^t(2t+1) \\ e^t(2t-1) \end{bmatrix}.$$

Superposition

A particular solution of

$$L[y] = e^{\sigma_1 t} v_1 + e^{\sigma_2 t} v_2 + \cdots + e^{\sigma_k t} v_k$$

is $y(t) = y_1(t) + y_2(t) + \cdots + y_k(t)$, where

$$L[y_1(t)] = e^{\sigma_1 t} v_1, \ldots, L[y_k(t)] = e^{\sigma_k t} v_1.$$

Prove this using linearity of L.

We next illustrate superposition by reworking

Example 7.6.2 of the text: Find a particular solution of

$$\frac{dy}{dt} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} y + \begin{bmatrix} 12 \\ 0 \end{bmatrix} e^{3t} + \begin{bmatrix} 0 \\ 18 \end{bmatrix} e^{2t}, \tag{6}$$

so that $\sigma_1 = 3, v_1 = \begin{bmatrix} 12 \\ 0 \end{bmatrix}$ and $\sigma_2 = 2, v_2 = \begin{bmatrix} 0 \\ 18 \end{bmatrix}$. Thus we are in Case No. 1 above. Thus

$$A - \sigma_1 I = \begin{bmatrix} -2 & 2 \\ 4 & 0 \end{bmatrix},$$

so that

$$(A - \sigma_1 I)^{-1} = \frac{1}{8} \begin{bmatrix} 0 & 2 \\ 4 & 2 \end{bmatrix},$$

and therefore

$$y_1(t) = -(A - \sigma_1 I)^{-1} v_1 = -e^{3t} \begin{bmatrix} 0 \\ 6 \end{bmatrix}.$$

For the second eigenvalue,

$$A - \sigma_2 I = \begin{bmatrix} -1 & 2 \\ 4 & 1 \end{bmatrix},$$

so that

$$(A - \sigma_2 I)^{-1} = \frac{1}{9} \begin{bmatrix} -1 & 2 \\ 4 & 1 \end{bmatrix},$$

and therefore

$$y_2(t) = -(A - \sigma_2 I)^{-1} v_2 = -e^{2t} \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

Combining,

$$y_p(t) = y_1(t) + y_2(t) = \begin{bmatrix} -4e^{2t} \\ -2e^{2t} - 6e^{3t} \end{bmatrix}.$$