

Undetermined coefficients special case: $ay'' + by' + cy = e^{\sigma t}$

We wish to solve

$$L[y] = g(t), \tag{1}$$

where

$$L[y] \doteq ay'' + by' + cy,$$

$a, b,$ and c are real and $g(t)$ has a special form to be discussed below. But first

Lemma 1 Let $y = \varphi + i\psi$ be a solution of

$$L[y] = g = g_1 + ig_2.$$

Then $L[\varphi] = g_1$ and $L[\psi] = g_2$.

Proof Use linearity. QED

Thus the solution of $L[y] = e^{\alpha t} \cos \beta t$ can be obtained by solving $L[z] = e^{(\alpha+i\beta)t}$ and then

$$y(t) = \operatorname{Re} z(t).$$

So if g in (1) has the form

$$e^{\alpha t} \cos \beta t \quad \text{or} \quad e^{\alpha t} \sin \beta t$$

we need only solve $L[y] = e^{\sigma t}$ where $\sigma = \alpha + i\beta$.

Note the following crucial identity

$$L[e^{rt}] = Z(r)e^{rt} \tag{2}$$

where $Z(r) \doteq ar^2 + br + c$ is the characteristic polynomial. There are three cases:

1. $Z(\sigma) \neq 0$, (σ not a root of Z): Motivated by (2) we look for a solution of

$$L[y] = e^{\sigma t}$$

in the form $y = ce^{rt}$. Thus, again using linearity,

$$L[ce^{rt}] = cZ(r)e^{rt} = e^{\sigma t}$$

if $r = \sigma$ and $c = \frac{1}{Z(\sigma)} = \frac{1}{a\sigma^2 + b\sigma + c}$. Thus we have the desired solution

$$L\left[\frac{e^{\sigma t}}{a\sigma^2 + b\sigma + c}\right] = e^{\sigma t}. \tag{3}$$

2. $\mathbf{Z}(\sigma) = \mathbf{0}$, $\mathbf{Z}'(\sigma) \neq \mathbf{0}$, (σ a simple root of Z):

Here, Z factors, $Z(r) = a(r - \sigma)(r - r_1)$, $r_1 \neq \sigma$. Differentiate (2) with respect to r to get

$$L[te^{rt}] = Z(r)te^{rt} + Z'(r)e^{rt} \quad (5)$$

and therefore

$$L[cte^{rt}] = cZ'(r)e^{rt} = e^{\sigma t}$$

provided $r = \sigma$ and $c = \frac{1}{Z'(\sigma)} = \frac{1}{a(\sigma - r_1)}$, i.e.

$$L\left[\frac{t}{a(\sigma - r_1)} e^{\sigma t}\right] = e^{\sigma t}.$$

3. $\mathbf{Z}(\sigma) = \mathbf{Z}'(\sigma) = \mathbf{0}$, $\mathbf{Z}''(\sigma) = 2\mathbf{a} \neq \mathbf{0}$, (σ a double root of Z): Apply $\frac{\partial}{\partial r}$ to (5) to get

$$L[t^2e^{rt}] = Z(r)[\text{STUFF}] + Z'(r)[\text{STUFF}] + Z''(r)e^{rt}$$

and therefore

$$L[ct^2e^{rt}] = cZ''(r)e^{rt} = e^{\sigma t}$$

if $r = \sigma$ and $c = \frac{1}{2a}$. Thus we have our solution

$$L\left[\frac{1}{2a}t^2e^{\sigma t}\right] = e^{\sigma t}.$$