The Kernel Method

For Ordinary Differential Equations with constant coefficients having the following form

$$y'' + by' + cy = g(t), \tag{1}$$

the Kernel Method is often much simpler to apply than Variation of Constants. We state it as a Theorem. (**NOTE:** The coefficient of y'' is 1.)

Theorem 1 (Kernel Method) *Let $K(t)$ be the unique solution of*

$$y'' + by' + cy = 0, \qquad y(0) = 0,\ y'(0) = 1. \tag{2}$$

Then

$$\phi(t) = \int_{t_0}^{t} K(t-s)g(s)ds$$

is the particular solution of (1) satisfying

$$y(t_0) = 0, \qquad y'(t_0) = 0.$$

Example: Find a particular solution of

$$y'' + y = \sec t \tan t\ . \tag{3}$$

Solution: Any solution of $y'' + y = 0$ has the form

$$y(t) = A\cos t + B\sin t.$$

Imposing the initial conditions in (2), we get $A = 0$ and $B = 1$. Thus $K(t) = \sin t$ and

$$\begin{aligned}
\phi(t) &= \int_0^t \sin(t-s)\sec s \tan s\ ds \\
&= \int_0^t [\sin t \cos s - \cos t \sin s]\ \frac{1}{\cos s}\ \frac{\sin s}{\cos s} ds \\
&= \sin t \int_0^t \tan s\ ds - \cos t \int_0^t \tan^2 s\ ds \\
&= \sin t \log(|\sec t|) - \cos t(\tan t - t)
\end{aligned}$$

is the particular solution of (3) satisfying $y(0) = y'(0) = 0$.