Coalitional Games for Transmitter Cooperation in Wireless Networks

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Abstract

Cooperation between nodes in a wireless network will increasingly become necessary to achieve performance goals when nodes employ the same resource spectrum. The question of feasibility of cooperation between *rational* nodes in a wireless network and whether there exists a fair division of the benefits of cooperation is unknown even for simple multi-terminal networks such as multiple access and broadcast channels, much less for a large wireless network. This paper addresses the stability of the grand coalition of transmitters (TXs) signaling over a multiple access channel using the framework of cooperative game theory. By modeling the game in *partition form*, each coalition’s utility can be accurately characterized by accounting for external interference. Single user decoding (SUD) and successive interference cancellation (SIC) receivers are considered and it is shown that the grand coalition maximizes the obtainable sum utility for both the decoding schemes. The grand coalition is stable if the core of the game, the set of all divisions of utility which cannot be achieved by any smaller coalition of TXs, is nonempty. For an SUD receiver, TX cooperation is shown to be stable at both high and low SNRs, while for an SIC receiver with a fixed decoding order, the cooperation is stable only at very low SNRs. However, for an SIC receiver which allows time-sharing between decoding orders, it is shown using an approximate utility function that TX cooperation is also stable at high SNRs. Finally, using numerical simulations, it is observed that the core is nonempty, in general, for an SUD receiver and an SIC receiver with time-sharing between decoding orders, thus showing that *ideal zero cost TX cooperation over a MAC channel improves achievable rates for each individual user in every coalition.*

I. Introduction

Next generation wireless networks are being designed to operate in a complex and dynamic environment in which nodes interact and cooperate to improve network throughput (see [3]–[8] and the references

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therein). As nodes signaling via wireless share resources due to the broadcast nature of the medium, cooperation between such nodes has emerged as a key strategy for improving performance [9]. In typical cooperative scenarios, it is inherently assumed that all nodes are controlled centrally and hence cooperation can be enforced. However, the emergence of heterogenous networks without central control challenges this assumption. In such scenarios, it is reasonable to assume that a rational node would willingly cooperate only if cooperation improves its own utility. The problem of determining which nodes in a network would cooperate in a stable fashion and how the benefits of such cooperation would be shared has thus become important especially in these future heterogenous networks. However, the feasibility and stability of cooperation is not completely understood even for simple multi-terminal networks such as multiple access (MAC) and broadcast channels, much less for larger networks.

We consider a staged approach to address the question of cooperative behavior among nodes using the framework of cooperative game theory. Our strategy is to first analyze the cooperative behavior in simple networks such as multiple access, broadcast, relay and interference channels etc. Once the properties of cooperation are well understood in small networks, a large network can be viewed as a composition of several elementary networks and cooperative behavior can then be examined to draw useful insights. To this end, this paper addresses the problem of feasibility and stability of cooperation among nodes signaling over a multiple-input multiple-output (MIMO) MAC.

A non-cooperative game between TXs in a MAC has been analyzed for various scenarios such as fading multiple-input single-output (MISO) channels in [10], for fading MIMO channels in [11] and for incomplete channel state information in [12]. To enhance the achievable throughput in a MAC, rational users (TXs here) can form coalitions and cooperate by signaling jointly to the receiver (see Fig. 1). To maximize the network throughput, there is an incentive to form the grand coalition (GC), the coalitions of all users, and signal jointly such that no data streams are decoded in the presence of interference. A larger coalition implies that the benefits of cooperation have to be shared among many users and the GC is stable only when no coalition of users has an incentive to break away and form a smaller coalition. The key question is to determine whether the gains of the GC are sufficient to share the payoff amongst all its members such that no coalition of TXs has an incentive to defect, thus ensuring stable cooperation. This is equivalent to checking the nonemptiness of the core, the feasible set of a linear program which describes the demands of each coalition. Stability of a coalition, i.e., the existence of a nonempty core imply that there exist signaling schemes in a practical network that can ensure cooperation.

Several works in the literature have examined the nonemptiness of the core for TX cooperation in a Gaussian MAC under various assumptions, primarily using characteristic form games (CFGs), in which
### Table I

<table>
<thead>
<tr>
<th>Receiver</th>
<th>Regime</th>
<th>Power Constraint Type</th>
<th>Core</th>
</tr>
</thead>
<tbody>
<tr>
<td>SUD</td>
<td>high SNR</td>
<td>SP and PP SP</td>
<td>Non-empty</td>
</tr>
<tr>
<td></td>
<td>low SNR</td>
<td>PP</td>
<td>Non-empty</td>
</tr>
<tr>
<td>SIC with fixed decoding order</td>
<td>high SNR</td>
<td>SP and PP SP</td>
<td>May be empty</td>
</tr>
<tr>
<td></td>
<td>low SNR</td>
<td>PP</td>
<td>Non-empty</td>
</tr>
<tr>
<td>SIC with time shared decoding orders</td>
<td>high SNR</td>
<td>SP and PP SP</td>
<td>Non-empty</td>
</tr>
<tr>
<td></td>
<td>low SNR</td>
<td>PP</td>
<td>Non-empty</td>
</tr>
</tbody>
</table>

Table showing the summary of the nonemptiness of the core of the TX cooperation game for various scenarios. SP = sum power constraint and PP = per-antenna power constraint.

The primary contribution of this paper is to study the problem of TX cooperation in a MAC under the framework of partition form game (PFG) theory [17] and show the stability of TX cooperation by proving the nonemptiness of the core in several scenarios of interest.

Specifically, we consider a MAC in which several TXs communicate simultaneously with an RX capable of performing either single user decoding (SUD) or successive interference cancellation (SIC). The TX cooperation game is modeled in two stages. First, for a given partition of the set of TXs, the Nash equilibrium (NE) rate point is determined and shown to be unique. The NE rate point can be treated as the operating point of the MAC channel with the current cooperation scenario. Next, coalitions in any given partition are allowed to merge to form the grand coalition. We show that for SUD and SIC receivers,
the TX cooperation game is *r*-super-additive and hence the grand coalition has the highest utility among all possible partitions. Next we show that the TX cooperation game has negative externalities for single antenna RX and mixed externalities for a multiple antenna RX. We consider the various cores of a PFG [18] and examine the stability of cooperation by investigating their nonemptiness. Table I shows a summary of results for the various scenarios considered in this paper. For an SUD receiver, the core is non-empty at both high and low SNRs, while for a SIC receiver with a given decoding order, the core is non-empty only in the noise-limited regime and may be empty at high SNR. This can be attributed to the asymmetry between the users caused by a fixed decoding order. However, if time sharing between decoding orders is permitted, we show, using an approximate utility function, that the core is again nonempty at high SNR, showing the stability of the GC. We note that while our results are true at high SNR for both sum power and per-antenna power constraints, the low SNR regime is characterized only for sum power constraints due to the lack of a suitable approximation of the capacity of a MIMO channel with per-antenna power constraints in this regime [19]. Numerical simulations show that the core of the TX cooperation game with SUD and time-shared SIC receivers (with approximate utility) is nonempty for all SNRs of interest, thus showing that the type of receiver and the SNR play a significant role in determining the stability of TX cooperation.

The rest of the paper is organized as follows. In Section II, we describe some preliminaries used in this paper. Section III describes the TX cooperation game for a SUD receiver and Section IV illustrates the same for an SIC receiver with fixed and time-shared decoding orders. Section V concludes this paper.

II. PRELIMINARIES

We begin by reviewing several game theoretic preliminaries for cooperative games. Let \( S \subseteq \mathcal{K}, \mathcal{K} = \{1, 2, \ldots, K\} \) denote an arbitrary coalition of TXs.

**Definition 1.** A partition \( T \) of \( \mathcal{K} \) is defined as a set of coalitions \( S_1, S_2, \ldots, S_N \) such that \( S_i \cap S_j = \emptyset, \forall i, j \in \{1, 2, \ldots, N\}, i \neq j \) and \( \bigcup_{i=1}^{N} S_i = \mathcal{K} \).

The set of all partitions of \( \mathcal{K} \) is denoted by \( \mathcal{T} \). The total number of partitions of a \( K \)-element set is called the Bell number \( B_K \), and satisfies the recursive relation \( B_{K+1} = \sum_{n=0}^{K} B_n, \) \( B_0 = 1 \) and \( B_K \) increases exponentially in \( K \).

In a cooperative game, players form coalitions and each coalition chooses an action from the set of actions jointly available to it (which may be larger than the set of actions available individually to all the players). By this choice of actions, each coalition \( S \) in partition \( T \) obtains a utility (value) denoted
Games in which the utility of each coalition is dependent on the actions of other coalitions are called *partition form games* (PFGs) while games in which utility is independent of external actions are called *characteristic form games* (CFG), i.e., \( v(S; T) \) is independent of the specific partition \( T \). Each coalition divides the utility its obtains among its players; we denote by \( x_{k,S} \) the utility obtained by the \( k \)th player in coalition \( S \).

**Definition 2.** A coalitional game is called a transferable utility (TU) game, if the cooperative gains achieved by a coalition can be arbitrarily divided among all members of the coalition.

**Definition 3.** A PFG is said to be cohesive if for any partition \( T = \{S_1, S_2, ..., S_N\} \) of \( K \),

\[
v(K; K) \geq \sum_{n=1}^{N} v(S_n; T). \tag{1}
\]

If a game is cohesive then the utility obtained by the grand coalition is larger than the sum of the utilities of each coalition under any partition.

**Definition 4.** A PFG is \( r \)-super-additive if for any disjoint coalitions \( S_1, S_2, ..., S_r \) and any partition \( \rho \) of \( K \setminus (S_1 \cup S_2 \cup ... \cup S_r) \),

\[
v(S_1 \cup ... \cup S_r; \{S_1 \cup ... \cup S_r\} \cup \rho) \geq \sum_{t=1}^{r} v(S_t; \{S_1, S_2, ..., S_r\} \cup \rho). \tag{2}
\]

When \( r = 2 \), the above definition reduces to the conventional definition of super-additivity in PFGs [18].

For CFGs, the utility of the coalition is independent of the rest of the partition and hence cohesiveness follows from the 2-super-additive property. However, for PFGs, 2-super-additivity does not result in cohesiveness due to the dependence of the utility on the rest of the partition. Using the notion of \( r \)-super-additivity, it is clear that if the game is \( r \)-super-additive for all feasible values of \( r \) and \( \rho = \phi \), then the game is cohesive.

**Definition 5.** A PFG is said to have negative externalities if for any mutually disjoint coalitions \( S_1, S_2 \) and \( S_3 \) and any partition \( \rho \) of \( K \setminus (S_1 \cup S_2 \cup S_3) \),

\[
v(S_3; \{S_3, S_1 \cup S_2\} \cup \rho) \leq v(S_3; \{S_1, S_2, S_3\} \cup \rho). \tag{3}
\]

A game with positive externalities is defined similarly with the inequality reversed. In other words, a game has negative (positive) externalities if a merger between two coalitions does not increase (decrease) the utility of all other coalitions. A game with mixed externalities exhibits both positive and negative
externalities for different coalitions or for different realizations of the game parameters.

III. TX COOPERATION GAME WITH SINGLE USER DECODING

A. Signal Model

Consider a scenario with $K$ TXs simultaneously transmitting to an $M$-antenna receiver. The link between any pair of TX and RX antennas is modeled by a flat fading channel with additive white Gaussian noise (AWGN) and the signal at the $m$th RX antenna can be expressed as $Y_m = \sum_{k=1}^{K} h_{m,k} X_k + Z_m$, $k = 1, ..., K$, $m = 1, ..., M$, where $h_{m,k}$ is the channel gain between the $k$th TX antenna and the $m$th RX antenna and $Z_m$ is additive white Gaussian noise (AWGN) with variance $N_0$. Throughout this paper, we assume that the transmitted signals $X_k$ are drawn from a Gaussian codebook.

Now consider a partition $T = \{S_1, S_2, ..., S_N\}$ of $K = \{1, 2, ..., K\}$ to be the set of coalitions formed by the cooperating TXs. Signals from each coalition of TXs can be modeled as coming from a $|S_n|$-input, $M$-output MIMO channel. In vector form, the received signal can be expressed as

$$Y_{M\times1} = \sum_{n=1}^{N} H_n X_n + Z_{M\times1},$$

where $H_n$ is the channel gain matrix modeling the channel between the $n$th coalition of TXs and the RX, and $X_n$ is the signaling vector for the coalition $S_n$ and $Z$ models the AWGN. We consider two cases for each coalition of players: (1) a transmit sum power constraint for each coalition and (2) a per-antenna power constraint.

Consider an SUD receiver which decodes a given coalition’s signal by treating all interfering signals from other coalitions as noise. The utility obtained by each coalition is defined as the maximum obtainable rate achievable by the TXs in the coalition, given the actions of all other coalitions, assuming an SUD receiver and the given power constraint. The utility obtained by each coalition is thus dependent on the actions of other coalitions in the partition and this obtained utility is the Nash equilibrium (NE) rate for each coalition. To accurately consider this dependence, we consider the TX cooperation game as a PFG in contrast to a CFG model considered previously (see [13], [14]). We first briefly illustrate the problem formulation in [14] to highlight the differences with our model.

B. Jamming CFG Model

In the jamming utility model in [14], the utility of a coalition $S$ is defined as the maximum obtainable rate assuming worst case interference from TXs in $S^c = \mathcal{K} \setminus S$.

$$v(S) = \min_{Q_{S^c}} \max_{Q_S} I(X_S; Y) = \min_{Q_{S^c}} \max_{Q_S} \log \left( \frac{|N_0 I + H_S Q_S H_S^H + H_{S^c} Q_{S^c} H_{S^c}^H|}{|N_0 I + H_S Q_S H_S^H|} \right)^{-1},$$

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Clearly $v(S)$ does not depend on the partition of the transmitters outside $S$. The diagonal entries of $Q_S$ and $Q_{Sc}$ are restricted to satisfy the power constraints (either sum power constraint or a per-antenna power constraint). Clearly, this is a very conservative definition of utility as in practice the coalition $S^c$ will not attempt to jam the transmissions from $S$. In fact, the TXs in $S^c$ would choose a signaling strategy to maximize their own rate. Under the jamming assumption, [14] shows that TX cooperation is cohesive, but is not super-additive. Finally, using a counterexample, [14] shows that there may not exist a stable division of the utility of the GC among the TXs and the game exhibits oscillatory behavior, i.e., TX cooperation is not stable and hence voluntary cooperation among TXs cannot be expected under the model for utility.

C. Non-Jamming PFG Model

We analyze the TX cooperation game in several stages: (1) For a fixed partition $T$ of TXs, the interaction between the coalitions is a non-cooperative game and the existence and uniqueness of the NE rate point of this game is proved. The uniqueness of the NE rate point ensures that every coalition in a given partition has a unique utility which it can expect. (2) Now, as coalitions in partition $T$ merge together, we show that the TX cooperation game is $r$-super-additive and has mixed externalities. (3) We then show that the core of this game is nonempty, thus ensuring that there exists a feasible division of GC utility such that no coalition has an incentive to deviate from the GC. This shows that voluntary TX cooperation can be expected in a MAC with heterogeneous nodes.

1) Non-cooperative Game Between Coalitions: Consider a partition $T = \{S_1, S_2, ..., S_N\}$ of $K$. The rate obtained by $S_n$, dependent on the partition $T$ and denoted by $v(S_n; T)$, can be expressed as

$$v(S_n; T) = \max_{Q_n} I(X_n; Y)|_{Q_{-n}}, \quad n = 1, 2, ..., N,$$

where $Q_{-n}$ is the signaling strategy of all the other coalitions in $T$ except $S_n$. Evaluating the mutual information term, we get

$$v(S_n; T) = \max_{Q_n} \left( \frac{|N_0I + H_nQ_nH_n^H + \sum_{j=1, j\neq n}^N H_jQ_jH_j^H|}{|N_0I + \sum_{j=1, j\neq n}^N H_jQ_jH_j^H|} \right)$$

From (7), we clearly observe that $v(S_n; T)$ depends on the signaling strategies (covariance matrices) of the other coalitions. We also use the notation $v_n(Q_n, Q_{-n})$ to highlight the dependence of the utility on the actions of all the players in the game. By definition, each coalition chooses an action which is the best response to the actions of the players in all the other coalitions and hence the utility $v(S_n; T)$ is a Nash equilibrium (NE) utility of the game.
2) Existence of Nash Equilibrium: The existence of the NE can be proved using the Kakutani fixed point theorem \([11], [20]\). We assume a per-antenna power constraint for illustration. Define the set

\[ A_n = \{ Q_n | Q_n \geq 0, [Q_n]_{k,k} \leq P_{n,k} \} \]

i.e., \( A_n \) is the set of all covariance matrices which satisfy the per antenna power constraint for the \( n \)th coalition. Clearly, \( A_n \) is a compact and convex set. Hence the set of \( N \)-tuples of covariance matrices defined by \( A = \times_{n=1}^{N} A_n \) is compact and convex. Now, define the function

\[ \Psi_n(Q_{-n}) = \left\{ Q_n \in A_n \left| v_n(Q_n, Q_{-n}) \geq v_n(\hat{Q}_n, Q_{-n}), \forall \hat{Q}_n \in A_n \right. \right\} \quad (8) \]

as the set of best responses by coalition \( S_n \) to the signaling strategy \( Q_{-n} \) adopted by other coalitions in the partition. Finally define \( \Psi : A \rightarrow A \) to be the set-valued function such that \( \Psi(Q_1, Q_2, ..., Q_N) = \times_{n=1}^{N} \Psi_n(Q_{-n}) \).

**Lemma 1.** The set of best responses \( \Psi_n(Q_{-n}) \) is convex.

The Lemma can be proved in a straightforward fashion using Lemma 3 in conjunction with the definition of \( B_n(Q_{-n}) \). Using Lemma 1, it can be shown that \( \Psi(Q_1, Q_2, ..., Q_N) \) is convex and the closure of \( \Psi(Q_1, Q_2, ..., Q_N) \) follows from the continuity of the utility function. The set of \( N \)-tuples of covariance matrices \( A \) and the function \( \Psi \) satisfy all the requirements of the Kakutani fixed point theorem \([20, p. 20]\) and hence there exists a fixed point of the set \( \Psi(Q_1, ..., Q_N) \) such that \( Q^*_n \in \Psi_n(Q^*_{-n}) \). The fixed point of this set guarantees the existence of a NE to the TX cooperation game with a non-jamming assumption.

3) Uniqueness of Nash equilibrium: The uniqueness of the NE for games with concave utility functions can be established by checking the (sufficient) condition for diagonally strict concavity (DSC) \([7], [21]\). The DSC condition can interpreted as the case where a player’s utility is more sensitive to the choice of his own actions as compared with the actions of all the other players. The form of the DSC condition as stated in \([7], [21]\) is well matched with the case where every player’s action variables are either scalars or vectors and hence cannot be applied directly to our scenario with matrix valued action variables. In \([11]\), the DSC condition is derived for a non-cooperative game between TXs in a MAC with sum power constraints for matrix valued strategies. However, the derived condition is problem specific and cannot be directly used herein where we have a per-antenna power constraint additionally. In addition, each coalition’s utility in \([11]\) is the ergodic capacity averaged over all channel realizations while our problem considers a fixed channel gain matrix. We derive a sufficient condition for the uniqueness of the NE for a game with per-antenna power constraints following the method in \([11], [21]\) and show that this is the
same condition as in [11] without the averaging over channel realizations.

**Proposition 1.** If \((\tilde{Q}_1, \tilde{Q}_2, ..., \tilde{Q}_N) \in A\) and \((\hat{Q}_1, \hat{Q}_2, ..., \hat{Q}_N) \in A\) be two sets of covariance matrices which are equilibria to the game and satisfying the condition \(v(\tilde{Q}_n, \tilde{Q}_-n) \neq v(\hat{Q}_n, \hat{Q}_-n)\), then

\[
C = \sum_{n=1}^{N} \text{Tr} \left\{ (\tilde{Q}_n - \hat{Q}_n)(\nabla_{Q_n} v_n(\hat{Q}_n, \hat{Q}_-n) - \nabla_{Q_n} v_n(\tilde{Q}_n, \tilde{Q}_-n)) \right\} \leq 0. \tag{9}
\]

**Proof:** By the definition of an NE, the covariance matrices are the solutions to the optimization problem in (7). The Lagrangian \(L_n\) for the maximization in (7) can be written as

\[
L_n = v_n(Q_n, Q_-n) + \text{Tr}(L_n Q_n) - \text{Tr}(D_n (Q_n - R_n)), \tag{10}
\]

where \(L_n\) is a positive semi-definite matrix, \(e_k\) is the canonical basis vector having 1 in the \(k\)th position and zeros at all other positions and \(D_n = \text{diag}(\lambda_1, ..., \lambda_{|S_n|})\) and \(R_n = \text{diag}(P_{n,1}, ..., P_{n,|S_n|})\) are the diagonal matrices containing the Lagrange multiplier coefficients and the power constraint values respectively. For non-trivial power constraints, the Slater condition is satisfied and the Karush Kuhn Tucker (KKT) conditions can be written as

1) \(\tilde{Q}_n \succeq 0, \hat{Q} \succeq 0\).
2) \(e_i^H \tilde{Q}_n e_i \leq P_{n,i}\) and \(e_i^H \hat{Q}_n e_i \leq P_{n,i}, i = 1, 2, ..., |S_n|\).
3) \(\text{Tr}(\tilde{L}_n \tilde{Q}_n) = 0\) and \(\text{Tr}(\hat{L}_n \hat{Q}_n) = 0\).
4) \(\text{Tr}(\tilde{D}_n (\tilde{Q}_n - R_n)) = 0\) and \(\text{Tr}(\hat{D}_n (\hat{Q}_n - R_n)) = 0\).
5) \(\nabla_{Q_n} v(\tilde{Q}_n, \tilde{Q}_-n) + \tilde{L}_n - \tilde{D}_n = 0\) and \(\nabla_{Q_n} v(\hat{Q}_n, \hat{Q}_-n) + \hat{L}_n - \hat{D}_n = 0\).

Now using the KKT conditions to evaluate and simplify \(C\), we get

\[
-C = \sum_{n=1}^{N} \text{Tr} \left[ (\tilde{Q}_n - \hat{Q}_n) \left( \nabla_{Q_n} v(\hat{Q}_n, \hat{Q}_-n) - \nabla_{Q_n} v(\tilde{Q}_n, \tilde{Q}_-n) \right) \right] \\
= \sum_{n=1}^{N} \text{Tr} \left[ (\tilde{Q}_n - \hat{Q}_n) \left( (\tilde{D}_n - \tilde{L}_n) - (\hat{D}_n - \hat{L}_n) \right) \right] \\
= \sum_{n=1}^{N} \text{Tr} \left[ \tilde{Q}_n \tilde{D}_n - \tilde{Q}_n \tilde{L}_n - \tilde{Q}_n \hat{D}_n + \tilde{Q}_n \hat{L}_n - (\tilde{Q}_n \tilde{D}_n + \tilde{Q}_n \tilde{L}_n + \tilde{Q}_n \hat{D}_n - \tilde{Q}_n \hat{L}_n) \right] \\
= \sum_{n=1}^{N} \text{Tr} \left[ R_n \tilde{D}_n - \tilde{Q}_n \tilde{D}_n + \tilde{Q}_n \hat{L}_n - \tilde{Q}_n \hat{D}_n + \tilde{Q}_n \hat{L}_n + R_n \tilde{D}_n \right] \\
\geq \sum_{n=1}^{N} \left[ \sum_{i=1}^{|S_n|} \lambda_i (P_{n,i} - e_i^H \tilde{Q}_n e_i) + \sum_{i=1}^{|S_n|} \lambda_i (P_{n,i} - e_i^H \hat{Q}_n e_i) \right] \geq 0. \tag{11}
\]
The sequence of equalities and inequalities directly follow from the KKT conditions. Proposition 1 shows that if there exist at least two equilibria of the non-cooperative game, then \( C \leq 0 \). Thus, a sufficient condition for the uniqueness of equilibrium is \( C > 0 \).

**Theorem 1.** The NE rate point of the TX cooperation game is unique.

**Proof:** Evaluating \( C \) for any two possible feasible strategies \((\hat{Q}_1, \hat{Q}_2, \ldots, \hat{Q}_N)\) and \((\hat{Q}_1, \hat{Q}_2, \ldots, \hat{Q}_N)\), we get

\[
C = \sum_{n=1}^{N} \text{Tr} \left[ (\hat{Q}_n - \hat{Q}_n)(\nabla_{Q_n} v_n(\hat{Q}_n, \hat{Q}_n) - \nabla_{Q_n} v_n(\hat{Q}_n, \hat{Q}_n)) \right]
\]

\[
= \text{Tr} \left[ \left( \sum_{n=1}^{N} H_n(\hat{Q}_n - \hat{Q}_n)H_n^{H} \right) \left( \left( I + \sum_{n=1}^{N} H_n\hat{Q}_nH_n^{H} \right)^{-1} - \left( I + \sum_{n=1}^{N} H_n\hat{Q}_nH_n^{H} \right)^{-1} \right) \right] \geq 0,
\]

(12)

where the last inequality is true from the fact that for positive definite matrices \( \hat{B} = I + \sum_{n=1}^{N} H_n\hat{Q}_nH_n^{H} \) and \( \hat{B} = I + \sum_{n=1}^{N} H_n\hat{Q}_nH_n^{H} \) we have \( \text{Tr} \left\{ (\hat{B} - \hat{B})(\hat{B}^{-1} - \hat{B}^{-1}) \right\} \geq 0 \) with equality only when \( \hat{B} = \hat{B} \).

From (7), we see that the utility function is determined by \( H_n\hat{Q}_nH_n^{H} \) and not the covariance matrices alone. Now, given any two feasible strategies for which \( H_n\hat{Q}_nH_n^{H} \neq H_n\hat{Q}_nH_n^{H} \) for some values of \( n \), we can clearly see that \( C > 0 \) and hence the NE rate point of the non-cooperative game between TX coalitions in a given partition is unique.

**Remark:** Fig. 3 shows the unique NE rate point for a scenario with 2 users and a single antenna receiver. We emphasize the fact that while the NE rate point is unique, the equilibrium achieving strategies may not be unique as there may exist distinct \( N \)-tuples of covariance matrices for which \( H_n\hat{Q}_nH_n^{H} = H_n\hat{Q}_nH_n^{H} \), \( \forall n \). As stated previously, the uniqueness of the NE rate point ensures that each coalition knows its utility uniquely. If the NE rate point were not unique, then the non-trivial question of which equilibrium rate point is to be chosen by all the players must be addressed.

4) **Evaluating the utility function:** For a coalition \( S_n \) in partition \( T \), the utility function as defined in (7) is the maximum achievable rate over the MIMO channel between the cooperating TXs and the receiver, given the interference of all other coalitions. For a coalition with sum power constraints, the capacity of the MIMO channel is given by a water filling solution. For the scenario with per-antenna power constraints, deriving a closed form solution remains an open problem in the literature and [19] evaluates the capacity in terms of the variables of the convex dual problem. However, for the special
case of a single antenna receiver \((M = 1)\), a closed form solution for capacity and hence coalition utility exists and is derived below.

The capacity of a MISO channel with sum power constraints and per-antenna power constraints is given as \(C^{SP} = \log \left(1 + \frac{1}{N_0} \left(\sum_{k=1}^{K} |h_m|^2 \left(\sum_{k=1}^{K} P_k\right)\right)\right)\) and \(C^{PP} = \log \left(1 + \frac{1}{N_0} \left(\sum_{k=1}^{K} |h_m|\sqrt{P_k}\right)^2\right)\) respectively \([19]\). Using the capacity results, the utility function for a single antenna receiver with per-antenna power constraints can be expressed as

\[
v(S_m; T) = \log \left(\frac{N_0 + \sum_{n=1}^{N} \left(\sum_{j \in S_n} |h_{j,n}|\sqrt{P_{j,n}}\right)^2}{N_0 + \sum_{n=1, n \neq m}^{N} \left(\sum_{j \in S_n} |h_{j,n}|\sqrt{P_{j,n}}\right)^2}\right),
\]

and with sum power constraints can be expressed as

\[
v(S_m; T) = \log \left(\frac{N_0 + \sum_{n=1}^{N} \left(\sum_{j \in S_n} |h_{j,n}|^2 \left(\sum_{j \in S_n} P_{j,n}\right)\right)}{N_0 + \sum_{n=1, n \neq m}^{N} \left(\sum_{j \in S_n} |h_{j,n}|^2 \left(\sum_{j \in S_n} P_{j,n}\right)\right)}\right).
\]

### D. Properties of the TX cooperation game

Previously, we have shown that for a given partition of TXs, there exists a unique NE rate point. Now, we illustrate the properties of the TX cooperation game in which coalitions in a partition cooperate with each other \((i.e.,\) merge together). Specifically, we show that the total utility obtained by the cooperating coalitions is larger than the sum of the utilities of the participating coalitions, \(i.e.,\) the game is \(r\)-super-additive. This implies that the GC of all TXs has the highest sum utility among all feasible partitions (even with externalities) of TXs and hence is the only feasible outcome of coalition formation. Next we show how cooperating coalitions influence the utility of the remaining coalitions in a partition.

**Proposition 2.** The TX cooperation game is \(r\)-super-additive and cohesive \((see (2))\), \(i.e.,\) for two partitions \(T_1 = (S_1, S_2, ..., S_r, S_{r+1}, ..., S_N)\) and \(T_2 = (S_1 \cup S_2 \cup ... \cup S_r, S_{r+1}, S_{r+2}, ..., S_N)\) of \(K,\) \(v(S_1 \cup ... \cup S_r; T_2) \geq \sum_{t=1}^{r} v(S_t; T_1)\) where all the utilities have been computed as in \((7)\).

**Proof:** Let \(\hat{Q} = (\hat{Q}_1, \hat{Q}_2, ..., \hat{Q}_N)\) and \(\hat{Q} = (\hat{Q}_{12...r}, \hat{Q}_{r+1}, ..., \hat{Q}_N)\) be the NE signaling covariance matrices for the partitions \(T_1\) and \(T_2\) respectively. This independent signaling reduces the search space over which the Nash equilibrium is computed for thecoalitions of interest as compared to joint signaling.
Thus,

\[
v(S_1 \cup \ldots \cup S_r; T_2) = [I(X_{S_1 \cup \ldots \cup S_r}, Y_K)]_{\tilde{Q}}^{(a)} \geq [I(X_{S_1 \cup \ldots \cup S_r}, Y_K)]_{\tilde{Q}}^{(b)} \geq \sum_{t=1}^{r} [I(X_{S_t}, Y_K)]_{\tilde{Q}} = \sum_{t=1}^{r} v(S_t; T_1),
\]

(15)

where (a) follows from the definition of the NE and the reduced search space for independent signaling; and (b) follows from the chain rule of mutual information and the fact that conditioning does not increase entropy. Hence, the TX cooperation game is \(r\)-super-additive. When all the coalitions cooperate, i.e., \(r = N\), the \(r\)-super-additivity property also implies cohesiveness.

\[\text{Proposition 3.} \quad \text{The TX cooperation game with a single antenna receiver has negative externalities, i.e., for two partitions} \ T_1 = (S_1, S_2, \ldots, S_r, S_{r+1}, \ldots, S_N) \text{ and} \ T_2 = (S_1 \cup S_2 \cup \ldots \cup S_r, S_{r+1}, S_{r+2}, \ldots, S_N) \text{ of} \ K, \ v(S_{r+i}, T_2) \leq v(S_{r+i}, T_1) \text{ for every} \ i = 1, \ldots, N - r \text{ and} \forall r.\]

\[\text{Proof:} \quad \text{The utility function for coalition} \ S_{r+i} \text{ for both the partitions can be expressed using (13) as}
\]

\[
v(S_{r+i}; T_2) = \log \left( \frac{N_{\text{int}} + (\sum_{j=1}^{r} a_j)^2 + a_{r+i}^2}{N_{\text{int}} + (a_1 + a_2 + \ldots + a_r)^2} \right)
\]

\[
v(S_{r+i}; T_1) = \log \left( \frac{N_{\text{int}} + \sum_{j=1}^{r} a_j^2 + a_{r+i}^2}{N_{\text{int}} + \sum_{j=1}^{r} a_j^2} \right),
\]

(16)

where \(a_n = \sum_{j \in S_n} |h_{j,n}| \sqrt{P_{j,n}} \geq 0 \) and \(N_{\text{int}} = N_0 + \sum_{j=1}^{N-r} a_{j+r}^2 > 0\). It is easy to see that \(v(S_{r+i}, T_2) \leq v(S_{r+i}, T_1) \) for every \(i = 1, \ldots, N - r\). The case of sum power constraints at the TX can be addressed similarly and hence the TX cooperation game with a single antenna receiver has negative externalities.

\[\text{Proposition 4.} \quad \text{The TX cooperation game with a multiple-antenna receiver has mixed externalities, i.e., some realizations of the game has positive externalities and other realizations have negative externalities.}\]

\[\text{Proof:} \quad \text{We demonstrate that the game has mixed externalities with an example. Consider a 3-user} \ (S_1, S_2, S_3) \text{ scenario each with a single antenna TX sending data to a 2 antenna RX. Let} \ h_{k} \text{ be the channel gain vector from the} \ k\text{th TX to the RX and let} \ P_k \leq 1 \text{ be the constraint on the transmit power for each antenna. Let} \ T_1 \text{ be a partition in which all the users are singletons and let} \ T_2 \text{ be a partition in which users 1 and 2 cooperate with each other. Now, the utility obtained by the 3rd user in partition} \ T_1 \text{ and} \ T_2 \text{ is different} \]

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can be expressed as

\[ v(S_3; T_1) = \log \left( \frac{|N_0 I + \sum_{i=1}^{3} h_i h_i^H|}{|N_0 I + \sum_{i=1}^{2} h_i h_i^H|} \right). \]  

(17)

In partition \( T_2 \), when users 1 and 2 cooperate with each other, the utility obtained by the 3rd user can be expressed as

\[ v(S_3; T_2) = \log \left( \frac{|N_0 I + H_{12} Q_{12}^{*} H_{12}^H + h_3 h_3^H|}{|N_0 I + H_{12} Q_{12}^{*} H_{12}^H|} \right), \]  

(18)

where \( H_{12} = [h_1, h_2] \) and \( Q_{12}^{*} \) is an NE of the non-cooperative game between the coalitions \( S_1 \cup S_2 \) and \( S_3 \) and is given as

\[ Q_{12}^{*} = \arg \max_{Q_{12}} \log \left( \frac{|N_0 I + H_{12} Q_{12} H_{12}^H + h_3 h_3^H|}{|N_0 I + h_3 h_3^H|} \right). \]  

(19)

It can be observed that for some realizations of the channel gains \( v(S_3; T_1) \leq v(S_3; T_2) \) and for other realizations \( v(S_3; T_1) > v(S_3; T_2) \). For example, when \( N_0 = 1 \), \( h_1 = [-1.148, -0.1049] \), \( h_2 = [0.7223, 2.5855] \) and \( h_3 = [-0.6669, 0.1873] \), we have that \( v(S_3; T_1) = 0.1911 \leq v(S_3; T_2) = 0.1930 \) while for \( h_1 = [0.1, -0.5445] \), \( h_2 = [0.3035, -0.6003] \) and \( h_3 = [0.4900, 0.7394] \) we have that \( v(S_3; T_1) = 0.508 \geq v(S_3; T_2) = 0.4803 \). Hence the TX cooperation game with a multiple antenna SUD receiver has mixed externalities in general.

The \( r \)-super-additive property for cooperating coalitions implies that the GC has the highest utility and hence is the only feasible outcome of cooperation. The GC is stable if and only if there exists a division of utilities such that no smaller coalition can improve on its allocated utility. To check the stability of the GC, we consider the core of a TU game, a linear program which describes the demands of each coalition given the behavior of users external to the coalition. The nonemptiness of the feasible set of this linear program implies the existence of an allocation of the GC utility such that no smaller coalition has an incentive to deviate from the GC. Stability of a coalition, i.e., the existence of a nonempty core implies that there exist signaling schemes in a practical network which can achieve the benefits of cooperation. While CFGs have a single definition of a core, several cores have been defined for PFGs to take into account the different behavior of remaining users [18]. We first state the definitions of the various cores and then investigate the nonemptiness of their feasible region.
E. Core of a PFG

The core of the game with rational expectations, named the $r$-core, is the feasible region for the set of linear inequalities:

$$\sum_{i \in S} x_i \geq v_{\rho^*_S}(S; \{S, \rho^*_S\}), \forall S \subseteq K, \sum_{i=1}^K x_i = v(K; K),$$

$$\rho^*_S = \arg \max_{\rho \in \rho} \sum_{G \in \rho} v(G; \{S, \rho\}).$$

The $r$-core models rational behavior among the remaining players i.e., all the other players excluding the deviating coalition try to maximize the sum utility, assuming that the deviating coalition cannot be changed anymore.

The core of the game with cautious expectations, named the $c$-core, is the feasible region for the set of linear equalities:

$$\sum_{i \in S} x_i \geq \min_{\rho} v(S; \{S, \rho\}), \forall S \subseteq K, \sum_{i=1}^K x_i = v(K; K),$$

where the minimization is carried out over all partitions $\rho$ of the remaining users $K \setminus S$. Let $\rho^*_S$ be the minimizing partition. Each coalition $S$ is guaranteed a reward of $v(S; \{S, \rho^*_S\})$, independent of the actual partition of the remaining users and the utility expected by each coalition is a conservative estimate of the actual obtainable utility.

The core of the game with merging expectations, named the $m$-core, is the feasible region for the set of linear inequalities:

$$\sum_{i \in S} x_i \geq v(S; \{S, K \setminus S\}), \forall S \subseteq K, \sum_{i=1}^K x_i = v(K; K).$$

Each coalition $S$ evaluates its utility by assuming that all the other players form a coalition irrespective of the actual partition of the users external to $S$.

Finally, the core with singleton expectations, named the $s$-core, as the feasible region of the set of inequalities:

$$\sum_{i \in S} x_i \geq v(S; \{S, (K \setminus S)\}), \forall S \subseteq K, \sum_{i=1}^K x_i = v(K; K),$$

where, $(K \setminus S)$ denotes the partition containing all the singletons. This is in direct contrast to the $m$-core where the utility of each coalition is computed by assuming that the rest of the users are in a single
coalition.

Fig. 2 shows an example of a nonempty core for a symmetric scenario. For PFGs with $r$-super-additivity, the rational behavior of external coalitions is to merge together and hence the $r$-core is identical to the $m$-core. For games with negative externalities, the utility of coalition is minimized when all other coalitions in a partition operate in a unified manner and hence the c-core is equal to the $m$-core in such games. Thus, for the TX cooperation game with a single antenna RX, the $r$-core, the $m$-core and the c-core are identical to each other while c-core might not be equal to the $m$-core for a game with a multiple antenna RX. The property of negative externalities for a game with a single antenna RX also implies that the constraints that define the s-core are tighter than the constraints that define the $m$-core and hence the s-core is a subset of the $m$-core. The nonemptiness of the various cores of a PFG is not guaranteed in general i.e., the stability of TX cooperation is not assured. Stability is important as it ensures that there always exists a feasible division of GC utility such that all coalitions have an incentive to participate in the GC. We now give the conditions under which the core of this game is nonempty.

F. Existence and Description of the Core

1) Necessary and Sufficient Conditions for Nonemptiness: For any $S \subseteq \mathcal{K}$, denote by $1_S \in \mathbb{R}^\mathcal{K}$ the characteristic vector of $S$ given by

$$(1_S)_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$

The collection $(\lambda_S), S \subseteq \mathcal{K}$ of numbers in $[0, 1]$ is a balanced collection of weights if for every player $k$, the sum of $(\lambda_S)$ over all the coalitions that contain the kth player is 1, i.e., $\sum_{S \subseteq \mathcal{K}} \lambda_S 1_S = 1_\mathcal{K}$. The Bondareva Shapley theorem [20, p.262] states that a CFG with transferable payoff has a nonempty core if and only if the game is balanced:

$$\sum_{S \subseteq \mathcal{K}} \lambda_S v(S) \leq v(\mathcal{K}).$$

(24)

Though the Bondareva Shapley theorem has been derived in the context of CFGs, it can be applied to PFGs to examine the nonemptiness of the various cores. It is well known that super-additive games may not satisfy the balancedness condition and can have empty cores [22]. Checking whether a game is balanced or not (i.e., verifying the nonemptiness of the core) is in general a co-NP-complete problem [23].

2) Sufficient Condition for Nonemptiness: A game with transferable payoff has a nonempty core if the utility function is supermodular. For example, the s-core of the TX cooperation game is nonempty if
for any $S_1, S_2, \subset \mathcal{K}$,
\[
v(S_1 \cup S_2; \{S_1 \cup S_2, (\mathcal{K} \setminus (S_1 \cup S_2))\}) + v(S_1 \cap S_2; \{S_1 \cap S_2, (\mathcal{K} \setminus (S_1 \cap S_2))\}) \\
\geq v(S_1; \{S_1, (\mathcal{K} \setminus S_1)\}) + v(S_2; \{S_2, (\mathcal{K} \setminus S_2)\}).
\] (25)

Having stated the necessary and sufficient condition for the nonemptiness of the core, we now show that the Bondareva Shapley theorem holds at the high SNR and the low SNR regime thus showing the TX cooperation game with SUD receivers has a nonempty core and proving that the GC is indeed stable.

**Theorem 2.** In the high SNR regime, i.e. $N_0 \to 0$, the TX cooperation game with an SUD receiver has a nonempty core.

**Proof:** Let us consider the s-core of the game. Every coalition $S$ computes its expected utility by assuming that all users outside the coalition act as a singleton. Let $(Q^*_S, Q^*_1, Q^*_2, \ldots, Q^*_{K-|S|})$ be the Nash equilibrium of the non-cooperative game with one coalition and all singleton users. For any balanced collection of weights $\lambda_S$ and as $N_0 \to 0$, we get
\[
\sum_{S \subseteq N} \lambda_S v(S; \{S, [\mathcal{K} - S]\}) = \sum_{S \subseteq N} \lambda_S \log \left( \frac{|N_0 I + H_S Q^*_S H^H_S + \sum_{j=1}^{K-|S|} H_j Q^*_j H^H_j|}{|N_0 I + \sum_{j=1}^{K-|S|} H_j Q^*_j H^H_j|} \right) \\
\leq (a) \log \left( \frac{|N_0 I + H_{\mathcal{K}} Q^*_{\mathcal{K}} H^H_{\mathcal{K}}|}{|N_0 I|} \right) = v(\mathcal{K}; \mathcal{K}),
\] (26)

where $(a)$ is true as the summation on the LHS has a finite value for large SNR and the RHS term increases in an unbounded fashion thus satisfying the conditions of the Bondareva Shapley theorem. Thus, the s-core of the game is nonempty. Using the exact same arguments, it can be shown that the various other cores defined for this game are nonempty as well. In addition, we note that the proof is independent of the nature of the power constraints and hence the theorem is true for the scenario with sum power constraints and per-antenna power constraints.

Before showing the nonemptiness of the core for the low SNR regime, we first evaluate the utility of coalitions at low SNR.

**Lemma 2.** In the low SNR regime, i.e., $N_0 \to \infty$, the capacity achieved by a player (here player 1) under sum power constraints and an SUD receiver can be approximated as
\[
v(Q_1, Q_{-1}) = \max_{\text{Tr}(Q_1) \leq P_1} \log \left( \frac{|N_0 I + H_1 Q_1 H^H_1 + K_{\text{intf}}|}{|N_0 I + K_{\text{intf}}|} \right) \approx \frac{\sigma_{H_1}^2 P_1}{N_0},
\] (27)
where $\sigma_{H_1}$ is the maximum singular value of $H_1$ and $K_{int} = \sum_{j=2}^{K} H_j Q_j^* H_j$ is the interference from all other users.

**Proof:** We begin by showing that at low SNR, the channel capacity is maximized by allocating all the power to the dominant eigen-mode of the channel.

$$v_1(Q_1, Q_{-1}) = \max_{Q_1} \left( \frac{|N_0 I + H_1 Q_1 H_1^H + K_{int}|}{|N_0 I + K_{int}|} \right) = \max_{Q_1} \left( \frac{|N_0 I + \tilde{H}_1 Q_1 \tilde{H}_1^H|}{|N_0 I|} \right),$$

$$\approx \max_{Q_1} \log \left( \frac{|N_0 I + U_1 \Sigma_1 V_1^H Q_1 V_1 \Sigma_1 U_1^H|}{|N_0 I|} \right) = \max_{\Tr(D) \leq P} \left( \frac{|N_0 I + \Sigma_1 D \Sigma_1|}{|N_0 I|} \right)$$

$$= \sum_i \log \left( 1 + \frac{\sigma_i^2 d_i}{N_0} \right) \overset{(a)}{=} \log \left( 1 + \frac{\sigma_{max}^2 P_1}{N_0} \right) \overset{(b)}{=} \frac{\sigma_{max}^2 P_1}{N_0}, \quad (28)$$

where $\tilde{H}_1 = (I + \frac{1}{N_0} K_{int})^{-1/2} H_1$ and $\tilde{H}_1 \approx H_1$ as $N_0 \to \infty$, (a) is true as allocating all the power to the dominant eigen-mode is the water-filling solution. Note that the capacity does not depend on any of the interferers in this regime and hence there are no externalities in this regime. 

**Theorem 3.** In the low SNR regime, i.e., $N_0 \to \infty$, the TX cooperation game with a sum power constraint has a nonempty core.

**Proof:** Before showing the nonemptiness of the core, we derive the relation between the joint utilities of cooperating TXs and the utility of each individual TX. Consider two cooperating users with channel gains $H_1$ and $H_2$. The channel gain matrix of the cooperating TX can be written as $H = [H_1 | H_2]$. Using the fact that $HH^H = H_1 H_1^H + H_2 H_2^H$, we get that $\sigma_H^2 \leq \sigma_{H_1}^2 + \sigma_{H_2}^2$ where $\sigma_H$, $\sigma_{H_1}$, and $\sigma_{H_2}$ are the maximum singular values of $HH^H$, $H_1 H_1^H$ and $H_2 H_2^H$ respectively.

Now assuming that there are $K$ TXs indexed by $\mathcal{K} = \{1, 2, ..., K\}$, the necessary and sufficient condition for the nonemptiness of the core is given by Bondareva-Shapley theorem from (24):

$$\sum_{S \subseteq \mathcal{K}} \lambda_S v_S \leq v_K \Rightarrow \sum_{S \subseteq \mathcal{K}} \lambda_S \frac{\sigma_{H_S}^2 P_S}{N_0} \leq \frac{\sigma_{H_K}^2 P_K}{N_0}, \quad (29)$$

where $v_S = \sigma_{H_S}^2 P_S / N_0$ is the utility of coalition $S$ from Lemma 2, $\sigma_{H_S}$ is the maximum singular value of the combined channel matrix of cooperating TXs $H_S$, $P_S = \sum_{i \in S} P_i$ and $\lambda_S$ is a balanced collection of weights. By substituting the upper bound on the singular values of an augmented matrix in (29) and comparing coefficients on both sides, it is easy to see that the Bondareva-Shapley theorem holds and hence the core of the TX cooperation game with sum power constraints is nonempty at low SNR. 

Theorem 2 and Theorem 3 show that the core of the TX cooperation game with SUD receivers is
nonempty in the high SNR and the low SNR regime. Showing the nonemptiness of the core for all SNRs has not proved to be a tractable problem, so far. However, we have observed that the various cores of this game are nonempty for all SNRs with extensive numerical simulations and a counterexample has not been found yet.

As previously stated, supermodularity of the utility function is a sufficient condition for the nonemptiness of the core. It can be shown with a counterexample that the TX cooperation game is not supermodular, in general, but there exist several specific instances where the game is supermodular. We briefly illustrate the relationship between $r$-super-additivity, supermodularity, balancedness and nonemptiness of the core with an example.

3) Example: Consider coalition formation over a 3-user MAC channel and study the $s$-core. For simplicity, denote by $v_S$, the utility of the coalition $S$ when the rest of the users act as singletons and $\lambda_S$ be a balanced set. Then the balancedness condition states that

$$
\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_{12} v_{12} + \lambda_{23} v_{23} + \lambda_{13} v_{13} + \lambda_{123} v_{123} \leq v_{123},
$$

$$
\lambda_1 + \lambda_{12} + \lambda_{13} + \lambda_{123} = 1, \lambda_2 + \lambda_{12} + \lambda_{23} + \lambda_{123} = 1, \lambda_3 + \lambda_{13} + \lambda_{23} + \lambda_{123} = 1, \lambda_S \geq 0 \forall S.
$$

Clearly, the set of all vectors $\lambda_S$ which satisfy the balancedness condition is a compact convex polyhedron. By the Caratheodory theorem [24], each point in the interior of a compact and convex set can be represented by a finite combination of its corner points. The corner points of the polyhedron for the 3-user case can be evaluated to be $(\lambda_1, \lambda_2, \lambda_3, \lambda_{12}, \lambda_{23}, \lambda_{13}, \lambda_{123}) = (1,1,1,0,0,0,0), (1,0,0,1,0,0,0), (0,1,0,0,0,1,0), (0,0,1,1,0,0,0), (0,0,0,0,0,0,1)$ and $(0,0,0,0,5,0.5,0.5,0)$. Thus it is necessary and sufficient to show that the balancedness condition holds at these 6 corner points. The inequalities which must hold true for the core of the 3-user game to be nonempty can then be expressed as

$$
1/2 (v_{12} + v_{23} + v_{31}) \leq v_{123}.
$$

Clearly, the first 4 inequalities are always satisfied as a result of $r$-super-additivity. For the last inequality to be true, it is sufficient (not necessary) that at least one of the following three conditions are satisfied.

$$
v_{12} + v_{13} \leq v_{123} + v_1, \ v_{12} + v_{23} \leq v_{123} + v_2, \ v_{23} + v_{13} \leq v_{123} + v_3.
$$

(31)
Clearly if the TX cooperation game is partially supermodular, i.e., if one of the above inequalities is true, then the balancedness condition is satisfied and the core of the game is nonempty. Thus, we see that supermodularity is a fairly strong requirement for the existence of the core. The set of corner points of the convex polyhedron characterizing $\lambda_S$ is called the minimal balanced set and the number of such points increases exponentially in the number of users. Finding all the corner points, verifying that the balancedness condition holds at each of these points and characterizing the degree of partial supermodularity required to ensure balancedness is an open problem in the literature.

G. Geometric Description of the Core

For general games with a nonempty core, finding a closed form expression for the corner points of the core in terms of the utility functions is non-trivial. However, when the game is supermodular, the corner points have a very simple description as function of the utility of the coalition. The definition of the contra-polymatroid for supermodularity is dependent on the assumed behavior of other coalitions. For the s-core, we define the contra-polymatroid as

**Definition 6.** Let $T = \{S_1, S_2, ..., S_N\}$ define a partition of $\mathcal{K}$ and $f : (S; T) \rightarrow R_+$ be a function defined for each coalition of every partition of $\mathcal{K}$. The polyhedron $G(f) = \{(x_1, x_2, ..., x_K) : \sum_{i \in S} x_i \geq f(S; \{S, (\mathcal{K} - S)\})\}$ is a contra-polymatroid if $f$ satisfies

1) $f(\phi, T) = 0$ (normalization)

2) $f$ is supermodular (see (25)).

For supermodular games, the dominant face of $G(f)$ is the core of the game. Let $\pi$ be any permutation of $\mathcal{K}$. Then the convex polyhedron defined by the vertices

$$
x_{\pi(1)} = f(\pi(1); \{\pi(1), \mathcal{K} - \pi(1)\})$

$$
x_{\pi(i)} = f(\{\pi(1), ..., \pi(i)\}; \{\{\pi(1), ..., \pi(i)\}, (\mathcal{K} - \{\pi(1), ..., \pi(i)\})\})$

$$
- f(\{\pi(1), ..., \pi(i-1)\}; \{\{\pi(1), ..., \pi(i-1)\}, \{\mathcal{K} - \{\pi(1), ..., \pi(i-1)\}\}\}) \tag{32}
$$

is the core of the PFG. For a given permutation $\pi$, the share of user $\pi(1)$ is the utility when $\pi(1)$ is a singleton coalition. The share of user $\pi(2)$ is the sum rate obtained jointly by the coalition $\{\pi(1), \pi(2)\}$ less the utility of $\pi(1)$ as a single user and so on. For the TX cooperation game, if the function $v$ is supermodular, then (32) provides an easy mechanism to calculate the share of each user for the given permutation $\pi$. Any point in the core can then be achieved by a convex combination of the corner points.
IV. TX COOPERATION GAME WITH SUCCESSIVE INTERFERENCE CANCELLATION

In Section III, we considered the problem of TX cooperation with SUD receivers. In this section, we consider a more powerful receiver which can perform successive interference cancellation (SIC) of the signals from each coalition and following the procedure in Section III, we investigate the stability of the TX cooperation.

Consider an arbitrary partition $T = \{S_1, S_2, ..., S_N\}$ of the available TXs. The received signal can be modeled as in the previous section by (4). The utility obtained by $S_n$ is now dependent on the signaling covariances matrices of every other coalition and the decoding order of the users at the receiver. As in Section III, the game is analyzed in several stages: (1) Knowing the current partition $T$, the receiver announces a decoding order for the coalitions in $T$ given by the permutation $\pi$ of $\{1, 2, ..., N\}$. (2) For $T$ and the announced decoding order, the game between the coalitions is a non-cooperative game and the existence and uniqueness of the NE is explored. (3) Assuming a fixed decoding order, the stability of TX cooperation is analyzed by assuming that coalitions cooperate and (4) Finally, SIC receivers which allow time-sharing between decoding orders are considered and stability of cooperation analyzed.

For a given partition $T$, when SIC is assumed at the receiver, the strategy of the kth coalition consists in choosing the best vector of precoding strategies (covariance matrices) $Q_k^{all} = (Q_k^{(1)}, Q_k^{(2)}, ..., Q_k^{(N)})$ each optimized for a given permutation $\pi$ of $\{1, 2, ..., N\}$. We first analyze the TX cooperation game for a given decoding order. Time-sharing between decoding orders is addressed later in this section.

Without loss of generality, assume that the decoding order of the coalitions is $\pi = (1, 2, ..., N)$. The utility of each coalition can then be expressed as

$$v(S_n; T) = \max_{Q_n} \log \left( \frac{|N_0 I + H_n Q_n H_n^H + \sum_{j=n+1}^{N} H_j Q_j^* H_j^H|}{|N_0 I + \sum_{j=n+1}^{N} H_j Q_j^* H_j^H|} \right).$$

(33)

By definition, each coalition adopts a signaling strategy which is the best response to the signaling strategy of the other coalitions and the decoding order announced by the receiver.

1) Existence of a NE: As in Section III, the feasible sets of each user are convex and compact and the utility functions are concave in each player’s action variable and continuous in all other variables. The existence of an NE follows from the Kakutani fixed point theorem.

2) Uniqueness of the NE rate point: Fig. 4 shows the SIC receiver for the decoding order $\pi = (1, 2, ..., N)$. The SIC imposes a unique structure on the utility functions which can be exploited to prove the uniqueness of the NE rate point. By examining the utility functions of coalitions in the reverse order
of decoding, we get
\[ v(S_N; T) = \max_{Q_N} \log \left( \frac{|N_0I + H_NQ_NH_N^H|}{|N_0I_K|} \right) \]
\[ v(S_{N-1}; T) = \max_{Q_{N-1}} \log \left( \frac{|N_0I + H_{N-1}Q_{N-1}H_{N-1}^H + H_NQ_NH_N^H|}{|N_0I_K + H_NQ_NH_N^H|} \right) \]
\[ \vdots \]
\[ v(S_1; T) = \max_{Q_1} \log \left( \frac{|N_0I + H_1Q_1H_1^H + \sum_{j=2}^{N} H_jQ_jH_j^H|}{|N_0I + \sum_{j=2}^{N} H_jQ_jH_j^H|} \right) \] (34)

Using the concavity of the log-det function in the leading variable, we see that \( v(S_N; T) \) has a unique maximum which may be achieved by multiple values of \( Q_N \). However, we note that the matrix \( H_NQ_NH_N^H \) is the same for each of the optimizing \( Q_N^* \). Now using this, we can show that \( v(S_{N-1}; T) \) has a unique maximum and so on. Computing this for all the coalitions in the reverse decoding order gives the unique NE rate point of the non-cooperative game played by the TXs. Alternatively, by using Proposition 1 we can establish the uniqueness of the NE.

Theorem 4. The NE rate point of the TX cooperation game with an SIC receiver and a given decoding order is unique.

Proof: Evaluating the DSC condition for any two feasible strategies \( \tilde{Q} = (\tilde{Q}_1, \tilde{Q}_2, ..., \tilde{Q}_N) \in A \) and \( \check{Q} = (\check{Q}_1, \check{Q}_2, ..., \check{Q}_N) \in A \), we get
\[
C = \sum_{n=1}^{N} \text{Tr} \left[ (\tilde{Q}_n - \check{Q}_n)(\nabla_{Q_n} v(\tilde{Q}_n, \tilde{Q}_n) - \nabla_{Q_n} v(\check{Q}_n, \check{Q}_n)) \right]
\]
\[
= \sum_{n=1}^{N} \text{Tr} \left[ (H_n\tilde{Q}_nH_n^H - H_n\check{Q}_nH_n^H) \right] \left\{ (N_0I + \sum_{j=n}^{N} H_j\tilde{Q}_jH_j^H)^{-1} - (N_0I + \sum_{j=n}^{N} H_j\check{Q}_jH_j^H)^{-1} \right\}
\]
\[
= \sum_{n=1}^{N} \text{Tr} \left[ (A_n - B_n) \right\{ \left( \sum_{j=1}^{n} B_j \right)^{-1} - \left( \sum_{j=1}^{n} A_j \right)^{-1} \}\right] \geq 0. \] (35)

where the matrices \( A_n \) and \( B_n \) are defined as \( A_1 = N_0I + H_N\tilde{Q}_nH_N^H, \ B_1 = N_0I + H_N\check{Q}_nH_N^H, \ A_{N-n+1} = H_n\tilde{Q}_nH_n^H \) and \( B_{N-n+1} = H_n\check{Q}_nH_n^H \) for \( n \geq 2 \). From [11], we know that the inequality is strict if \( A_n \neq B_n \) for at least one value of \( i \). Hence, for covariance matrices such that \( H_n\tilde{Q}_nH_n^H \neq H_n\check{Q}_nH_n^H \) for at least one value of \( n \), we see that \( C > 0 \). As the utility functions depend only on the values of \( H_NQ_NH_N^H \) and not the covariance matrices \( Q_n \) themselves directly, the NE rate point for the TX cooperation game is unique.

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Remark: Fig. 6 shows the NE rate point for an example scenario. The NE rate points are the corner points of the polymatroidal rate region of the MAC. From (34) we see that the NE rate point can be computed using sequential iterative water filling [25]. As in the previous section, we emphasize that the NE rate point is unique but there could be multiple $Q_n$’s which achieve this point. The uniqueness of the rate point implies that, given the decoding order, each TX can compute without ambiguity the equilibrium rate it can achieve.

A. Properties of the TX Cooperation Game

Cooperation between TXs (coalitions of TXs) over the MAC channel with an SIC receiver has two benefits: (1) Cooperating coalitions signal jointly which can result in an improvement in the achievable sum rate. (2) The decoding order of the combined coalition improves relatively in comparison to the decoding order of its member coalitions resulting in a further improvement in the achievable rate for the cooperating coalitions. This can be illustrated clearly with an example. Consider a MAC scenario with 4 TX coalitions $S_1, S_2, S_3, S_4$ specified in the order in which they are decoded. Assuming that coalitions $S_1$ and $S_3$ cooperate with each other, the receiver first decodes coalition $S_2$. Next $S_1$ and $S_3$ which signal jointly are decoded and $S_4$ is decoded in the final step. Clearly, $S_1$ benefits by moving later into the decoding order and both $S_1$ and $S_3$ benefit by signaling jointly. On the other hand, the utility of $S_2$ decreases as it is decoded first with interference from $S_1$ and $S_3$ jointly. Informally this shows that the TX cooperation game is r-super-additive and has negative externalities, i.e., cooperating coalitions benefit and cooperation decreases utility for others. As in Section III, super-additivity implies that the grand coalition has the maximum sum utility among all possible partitions of TXs and negative externalities induces TXs to form larger and larger coalitions. Before examining the stability of the grand coalition, we formalize the intuition provided in this paragraph.

Proposition 5. The TX cooperation game with SIC processing at the receiver is r-super-additive and cohesive, i.e., for two partitions $T_1 = (S_1, S_2, S_3, ..., S_r, S_{r+1}, ..., S_N)$ with decoding order $\pi_1 = (1, 2, ..., N)$ and $T_2 = (S_{a_1}, ..., S_{a_t}, S_{b_1} \cup S_{b_2} \cup ... \cup S_{b_r}, S_{t+r+1}, S_{t+r+2}, ..., S_N)$ with decoding order $\pi_2 = (a_1, a_2, ..., a_t, b_{(12...r)}, t + r + 1, ..., N)$

$$\nu(S_{b_1} \cup S_{b_2} \cup ... \cup S_{b_r}; T_2) \geq \sum_{i=1}^{r} \nu(S_{b_i}; T_1),$$

where all the utilities are as computed in (33). Note that $(a_1, ..., a_t, b_1, ..., b_r)$ is a permutation of $(1, 2, ..., t + r)$ satisfying $a_1 < a_2 < ... < a_t$ and $b_1 < b_2 < ... < b_r$.  

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Proof: Let $\hat{Q}$ and $\check{Q}$ be the NE achieving covariance matrix tuples of $T_1$ and $T_2$ respectively. Then,

$$v(S_{b_1 \cup \ldots \cup b_r}; T_2) = I(X_{b_1 \cup \ldots \cup b_r}; Y|X_{a_1}, \ldots, X_{a_t})|_{\hat{Q}}$$

$$(a) \geq I(X_{b_1 \cup \ldots \cup b_r}; Y|X_{a_1}, \ldots, X_{a_t})|_{\check{Q}}$$

$$(b) = I(X_{b_1}; Y|X_{a_1}, \ldots, X_{a_t})|_{\check{Q}} + I(X_{b_2}; Y|X_{a_1}, \ldots, X_{a_t}, X_{b_1})|_{\check{Q}} + \ldots$$

$$+ I(X_{b_r}; Y|X_{a_1}, \ldots, X_{a_t}, X_{b_1}, X_{b_2}, \ldots, X_{b_{r-1}})|_{\check{Q}}$$

$$(c) \leq \sum_{i=1}^{r} I(X_{b_i}; Y|X_1, X_2, \ldots, X_{b_{i-1}})|_{\check{Q}} = \sum_{i=1}^{r} v(S_{b_i}; T_1),$$

where the inequality $(a)$ follows from the assumption of independent signaling among the cooperating coalitions and the definition of the NE, $(b)$ and $(c)$ follow from the properties of mutual information and the fact that $(a_1, \ldots, a_t, b_1, \ldots, b_r)$ is a permutation of $(1, 2, \ldots, t + r)$ such that $a_1 < a_2 < \ldots < a_t, b_1 < b_2 < \ldots < b_r$. Hence the TX cooperation game with SIC is $r$-super-additive. Clearly, when all the coalitions cooperate, the TX cooperation game with SIC is cohesive.

Proposition 6. The TX cooperation game with a single antenna SIC receiver has negative externalities.

Proof: Let us assume that the definitions of the partitions and the decoding orders are as in Proposition 5. We first note that

$$v(S_{t+r+i}; T_2) = v(S_{t+r+i}; T_1), \; \forall i = 1, 2, \ldots, N - t - r.$$  \hspace{1cm} (38)

The statement is true as the utility function of coalition $S_{t+r+i}$ only depends on the undecoded coalitions in their respective partitions, which are identical for $T_1$ and $T_2$ (see Eq. 33). For the other coalitions, we first consider the proof for a single antenna receiver.

$$v(S_{a_n}; T_2) = \log \left( \frac{N_{\text{int}} + \alpha_n^2 + \left( \sum_{i=1}^{r} \alpha_{t+i} \right)^2}{N_{\text{int}} + \left( \sum_{i=1}^{r} \alpha_{t+i} \right)^2} \right)$$

$$v(S_{a_n}; T_1) = \log \left( \frac{N_{\text{int}} + \alpha_n^2 + \sum_{i=1}^{r} \alpha_{t+i}^2}{N_{\text{int}} + \sum_{i=1}^{r} \alpha_{t+i}^2} \right),$$  \hspace{1cm} (39)

where $n = 1, 2, \ldots, t$, $\alpha_n = \sum_{j \in S_n} |h_{j,n}|^2 \sqrt{P_j}$, and $N_{\text{int}} = N_0 + \sum_{i=n+1}^{t} \alpha_i^2 + \sum_{i=t+r+1}^{N} \alpha_i^2$. From the above expressions, it can be clearly seen that $v(S_{a_n}; T_2) \leq v(S_{a_n}; T_1)$ and hence the TX cooperation game with a single antenna SIC receiver has negative externalities.

Proposition 7. The TX cooperation game with a multiple antenna SIC receiver has mixed externalities.
Proof: We demonstrate using the example that the TX cooperation game with a multiple antenna receiver has mixed externalities. As in Proposition 4, consider a scenario with 3 users \((S_1, S_2, S_3)\) each with a single antenna TX transmitting to a 2-antenna RX. Let \(h_k\) be channel gain vector from the kth TX to the RX and let \(P_k \leq 1\) be the per-antenna power constraint for each TX antenna. Let \(T_1\) be the partition in which all the users are singletons and let \(T_2\) be the partition in which users 1 and 2 cooperate with each other. Let the decoding order for partition \(T_1\) be \(3 \rightarrow 2 \rightarrow 1\). If users 1 and 2 cooperate to form partition \(T_2\) then the decoding order for \(T_2\) is \(3 \rightarrow 12\). Now, the utility obtained by user 3 under the partitions \(T_1\) and \(T_2\) is given by (17) and (18) respectively with \(Q^*_{12} = \arg \max_{Q_{12}} \log \left( \frac{|N_0 I + H_{12} Q_{12} H_{12}^H|}{|N_0 I|} \right)\), instead of the expression for \(Q^*_{12}\) in (19). It can be observed that for some realizations of the channel gains \(v(S_3; T_1) \leq v(S_3; T_2)\) and for other realizations \(v(S_3; T_1) > v(S_3; T_2)\). For example, when \(N_0 = 1\), \(h_1 = [1.17119, -0.1941]\), \(h_2 = [-2.1384, -0.8396]\) and \(h_3 = [1.3546, -1.0722]\), we have that \(v(S_3; T_1) = 0.8580 \leq v(S_3; T_2) = 1.0023\) while for \(h_1 = [-1.5771, 0.5080]\), \(h_2 = [0.2820, 0.0335]\) and \(h_3 = [-1.3337, 1.1275]\) we have that \(v(S_3; T_1) = 0.7593 \geq v(S_3; T_2) = 0.7462\). Hence the TX cooperation game with a multiple antenna SIC receiver has mixed externalities in general.

B. Stability of the GC

As stated earlier, \(r\)-super-additivity implies that the GC is the only sum utility maximizing partition. We now examine the stability of TX cooperation for a MAC with an SIC receiver implementing a fixed decoding order.

Theorem 5. In the low SNR regime, i.e., \(N_0 \to \infty\), the TX cooperation game with a sum power constraint has a nonempty core.

Proof: Consider the case of sum power constraints for each TX coalition. From Lemma 2, we know that the capacity at very low SNRs can be considered independent of the interference experienced by the coalition under consideration. Following the exact methodology in Proposition 3, we can show that the TX cooperation game has a nonempty core at low SNRs.

Theorem 6. In the high SNR regime, i.e., \(N_0 \to 0\), the TX cooperation game may have an empty core.

Proof: We give an example to show that the game has an empty core at high SNR. Consider a 4-user MAC (K=4) with one antenna at each TX and the RX. The power constraint on each user is
and the noise variance is \( N_0 = 1 \) with each user having an identical channel gain of \( h_i = 1 \). The receiver performs SIC by decoding the users in the order specified by a permutation \( \pi \). Using the above parameters and computing the utilities as in (33), it can easily be verified that the s-core of this game is empty. By symmetry of the channel gains and power constraints on each user, the s-core of this game is empty for all decoding orders. It can be similarly observed that the c-core, the m-core and r-core for this game are also empty at high SNR.

In general, it is observed that the various cores of the TX cooperation game with SIC receivers can be empty at moderate to large SNRs for all decoding orders. Fig. 5 shows a plot of the boundary between the regions of empty and nonempty cores as a function of the number of players and the SNR (Each player is assumed to have unit maximum power constraint and unit channel gain and SNR is defined as \( 1/N_0 \), i.e., the scenario where players are identical in all aspects other than the decoding order). Clearly, the core of the game is nonempty at low SNR and is empty at higher SNRs. The empty core at high SNR can be attributed to the asymmetry between the TXs caused by a fixed decoding order. In the low SNR regime, the noise completely dominates the interference and the decoding order becomes irrelevant. This removes the asymmetry between the players and the core is nonempty this regime.

We next consider the scenario in which time-sharing is permitted between various decoding orders. From Theorem 5, the core for an SIC receiver is nonempty at low SNRs for a fixed decoding order and hence would be nonempty for a time shared SIC receiver. We now investigate the nonemptiness of the core at high SNRs.

C. SIC receiver with time sharing between decoding orders

Let \( \Theta \) be the set of all probability distributions characterizing the time sharing of decoding orders. The set \( \Theta \) is a convex polyhedron whose corner points are distributions which assign probability 1 to one of the decoding orders. Clearly all the elements of \( \Theta \) do not contribute to nonempty cores. Computing the subset of \( \Theta \) for which the core is nonempty appears to be intractable. Determining whether there exists a probability distribution of decoding orders for a given value of channel gains and power constraints such that the core is nonempty is also challenging. We thus consider a simplified game in which the utility function of a coalition is approximated by the dominant term and the rest of the terms which are decoded with interference are ignored. At high SNR, the dominant term can be considered a good approximation to the actual utility as the dominant term increases unboundedly while the other terms have a finite value. This approximation implies that the game is now a CFG. We now evaluate the core of this game with approximate utilities.
Theorem 7. The core of the TX cooperation game for an SIC receiver with equal probability of time sharing between all decoding orders is nonempty at high SNR to a first order approximation of utility.

Proof: For the scenario in which all the decoding orders have equal probability, we first compute the utility function of a given coalition of players. We note that the dominating term in the utility function is the term in which a given coalition $S$ is decoded without any interference and increases unboundedly at large SNR. All the other terms are bounded at high SNR due to the interference present while decoding and can be ignored in this approximate analysis. The utility for coalition $S$ averaged over all decoding orders can then be evaluated as

$$v_S = \frac{|S|}{K} \log \left( \frac{|N_0I + H_S Q_S^* H_S^H|}{|N_0I|} \right),$$

where $|.|$ for sets denotes the number of elements in it. The first order approximation of utility can be interpreted as the scaled (by $|S|/K$) sum rate obtained when coalition $S$ signals with no interference. This also implies that the approximated utility does not depend on the strategies of other coalitions. Substituting the utility in the Bondareva Shapley theorem in (24), the necessary and sufficient condition to be satisfied can be expressed as

$$\sum_{S \subseteq \mathcal{K}} \lambda_S \frac{|S|}{K} \log \left( \frac{|N_0I + H_S Q_S^* H_S^H|}{|N_0I|} \right) \leq \log \left( \frac{|N_0I + H_K Q_K^* H_K^H|}{|N_0I|} \right),$$

for some balanced set of numbers $\lambda_S$. The above condition is satisfied as the vector $\lambda_S |S|/K$ is a probability distribution and the utility for a coalition $S$ is always smaller than the utility of the GC. This ensures the core of the TX cooperation game with time sharing SIC receivers is nonempty at high SNR.

Numerical simulations using a first order approximation of utility show that when all the decoding orders are equally likely, the core is nonempty at all SNRs of interest showing that, in symmetrized situations, the GC of the TX cooperation game is stable and there exists a stable division of cooperation benefits such that no smaller coalition has an incentive to deviate.

A summary of the results in this paper is shown in Table I. We see that when interference was ignored in decoding, there is a strong incentive to form the GC wherein the messages of all the users are jointly decoded and TX cooperation is stable. However, when the receiver performs SIC in a fixed order, some users benefit more than the others and cooperation is stable only at very low SNRs where the effect of interference can be ignored and users cooperate to combat noise. When time sharing between decoding
orders is permitted, we see using an approximate utility function that TX cooperation is stable at high SNRs. A comprehensive characterization of the stability of cooperation in this scenario, especially with accurate utility functions, is still an open problem.

D. Games with empty cores

As seen earlier, the game can have an empty core and thus the GC has the highest utility but is not stable and may exhibit oscillatory behavior [14]. Several approaches have been suggested in the literature to ensure stability of cooperation. We consider the $s$-core for illustration. The $\epsilon$-core of a game is defined as the set of imputations such that

$$\sum_{i \in S \subset K} x_i \geq \min(v(S; \{S, [K - S]\}) - \epsilon, 0), \quad \sum_{i=1}^{K} x_i = v(K; K).$$

(43)

In effect, the receiver penalizes each coalition for leaving the GC. By choosing a large enough value of $\epsilon$, the $\epsilon$-core can always be made nonempty. The least core of the game is defined as the $\epsilon$-core for the smallest value of $\epsilon$ that makes the core nonempty. The least core can be obtained by solving the optimization problem

$$\epsilon^* = \min \epsilon, \text{ subject to } \sum_{i \in S \subset K} x_i \geq \min(v(S; \{S, [K - S]\}) - \epsilon, 0), \quad \sum_{i=1}^{K} x_i = v(K; K).$$

(44)

If cooperation is not stable, the receiver can penalize each deviating coalition to the extent determined by $\epsilon^*$ by refusing to decode signals sent at a higher rate and thus enforcing the stability of the GC.

V. Conclusions

The question of feasibility of cooperation between rational nodes in a wireless network and whether there exists a fair division of the benefits of cooperation is unknown even for simple multi-terminal networks such as MAC and broadcast channels, much less for a large wireless network. This paper addresses the problem of TX cooperation for a MAC using partition form cooperative game theory to accurately model the effects of interference. The stability of the grand coalition, the coalition of all transmitters, for SUD and SIC receivers is examined. For an SUD receiver, TX cooperation is shown to be stable at high and low SNRs analytically and at all SNRs numerically, while for an SIC receiver with a fixed decoding order, TX cooperation is only stable at low SNRs where interference is negligible. However, at high SNRs with an approximate utility function, TX cooperation is stable with an SIC receiver implementing equal time sharing between decoding orders. In summary, our work demonstrates
that under the assumption of zero costs, voluntary cooperation is feasible and stable between users in a MAC and every user benefits from cooperation assuming that the right receiver is selected based on the SNR regime.

**APPENDIX**

**Lemma 3.** *(from [26]*) The function \( f \) defined by

\[
f(K_x, K_z) = \log \left( \frac{|K_x + K_z|}{|K_z|} \right) = \log \left( |I + K_x K_z^{-1}| \right)
\]

is convex in \( K_z \) given \( K_x \) is symmetric positive semi-definite and strictly convex when \( K_x \) is positive definite. It is also strictly concave in \( K_x \) given \( K_z \) is symmetric positive definite.

**REFERENCES**


Fig. 1. Transmitters cooperate to form coalitions. All transmitters in a coalition fully cooperate with each other.


Fig. 2. Example of a nonempty core for a symmetric scenario.

Fig. 3. Figure showing the NE rate point for an SUD single antenna receiver.

Fig. 4. Decoding in an SIC receiver

Fig. 5. Plot showing the boundary of the region between empty and nonempty core as a function of SNR and the number of players for a symmetric scenario and a single antenna SIC receiver with a fixed decoding order.

Fig. 6. Figure showing the NE rate points for an SIC single antenna receiver for different decoding orders.