A Learning Scheme for Blackwell’s Approachability in MDPs and Stackelberg Stochastic Games

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The notion of approachability was introduced by Blackwell ([8]) in the context of vector-valued repeated games. The famous ‘Blackwell’s approachability theorem’ prescribes a strategy for approachability, i.e., for ‘steering’ the average vector-cost of a given player towards a given target set, irrespective of the strategies of the other players. In this paper, motivated from the multi-objective optimization/decision making problems in dynamically changing environments, we address the approachability problem in Markov Decision Processes (MDPs) and Stackelberg stochastic games with vector-valued cost functions. We make two main contributions. Firstly, we give simple and computationally tractable strategy for approachability for MDPs and Stackelberg stochastic games. Secondly, we give reinforcement learning algorithms to learn the approachable strategy when the transition kernel is unknown. We also show that the conditions that we give for approachability are both necessary and sufficient for convex sets and thus a complete characterization. We also give sufficient conditions for non-convex sets.

Key words: Blackwell’s Approachability, Markov Decision Processes (MDPs), Stochastic Games, Reinforcement Learning, Multi-objective Optimization.

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1. Introduction Classical game theory, Markov Decision Processes (MDPs) and stochastic games typically deal only with scalar performance criteria: corresponding to each state and action of the agents, each agent incurs a scalar cost. The standard problem is to compute or learn a policy for each agent which will minimize her scalar performance objective conditioned on the policies of all other agents and the dynamics of the underlying system. Computing or learning equilibrium policies in a standard normal form strategic game or stochastic game, computing or learning the optimal policy of average or discounted cost MDPs are the typical examples. However, many interesting problems often fall outside this ‘scalar performance criteria’ class. For example consider the problem of an automaker. They want to minimize their cost but also worry about reliability, perceived quality and customer satisfaction - all of which are quantified in some way, and should be greater than some prescribed values. Bandwidth allocation in a wireless network involves throughput maximization while also providing certain delay guarantees.

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Multi-objective optimization is a well studied area [17] though most methods focus on achieving a Pareto-optimal solution. In the context of decision making under dynamically changing environment, these problems have been studied extensively under the class ‘constrained MDPs’ [2]. A reinforcement learning algorithm for constrained MDPs was developed in [10]. [18] and [12] consider MDPs with arbitrary reward process. Their setting is in the framework of regret minimization which is different from our approach.

In this paper, we address the problem of multi-objective optimization in a dynamically changing environment in the context of MDPs with vector-valued cost functions. Our objective is to learn the control policies that will drive the average vector-cost to a given ‘target set’. The problem formulation is different from that of constrained MDPs: the constrained optimization framework is replaced by a constraint satisfaction framework. This problem formulation is inspired from the famous ‘Blackwell’s approachability theorem’ [8]. In his work, Blackwell posed the following question: does there exist a strategy in an arbitrary two-player game with vector-payoffs which guarantees that player 1’s average payoff approaches (with probability 1) a given closed convex set \( D \) irrespective of the other player’s strategy? Blackwell characterized the games for which there is an affirmative answer and he also prescribed a strategy which achieves this.

We note that while the approachability question has mostly been asked in the context of games, with competing decision makers, more basically, it is about multiple objectives. Approachability in a stochastic game framework has been addressed before. [16] studied this problem where the approachability from a given initial state was studied under some recurrence assumptions. Their scheme depends on updating strategies when the system returns to a fixed state \( s_0 \). This scheme was proposed because there appeared to be a need to keep the policy fixed for some duration in order to ‘exploit’ that policy before one ‘explores’ again. However, there are many computational difficulties associated with this approach, in particular the return time to a fixed state. Firstly, this approach has the undesirable effect of increasing the variance of the cost for the agents if the recurrence times are large, e.g., in very large systems. For this reason, the scheme will have slow convergence. [15] proposed an alternative scheme that required less restrictive assumptions. The basic idea is the ‘increasing time window’ method: keep the policy constant for the length of a time window whose duration increases gradually. In each window, the policy used is the equilibrium policy of an \( N_i \)-stage stochastic game where \( N_i \) is the length of the \( i \)th window. The computation of this policy is thus clearly non-trivial. [13] presents yet another scheme which has the same drawbacks. These schemes also did not address another important aspect of this problem: a learning algorithm for approachability when the transition kernel corresponding to the underlying Markov dynamics is unknown. [14] gave a learning algorithm that builds on the work in [16]. The key idea is to run \( J \) learning algorithms in parallel, each one corresponds to a different steering direction. If \( J \) is sufficiently large, \( \epsilon \)-approachability can be guaranteed. This scheme however is potentially computationally impractical in some scenarios.

In this paper, we make the following contributions. We first consider Markov decision process with vector cost functions. We give necessary and sufficient conditions for approachability of convex sets. We also give sufficient conditions for approachability of non-convex sets. Some of these conditions are similar to prior work but the proof is entirely different as we rely on ideas from stochastic approximation theory for constructing an approachability strategy. It turns that it is easy to construct a ‘learning’ scheme, i.e., when the model parameters are unknown, from our approachability strategy. We then consider Stackelberg stochastic games. These are of course special stochastic games. We give necessary and sufficient conditions for approachability of convex sets and sufficient conditions for non-convex sets. In that sense, our approachability results are weaker since other prior work [16, 15, 13], all consider general stochastic games. However, again using multiple time scale stochastic approximation theory, we are able to give a ‘learning’ scheme for Stackelberg stochastic games. It seems difficult to derive a learning scheme from approachability strategies for stochastic games from any of the prior works (except [14]). Thus, our main
contributions are: (i) new proofs for approachability for MDPs and Stackelberg stochastic games, and (ii) stochastic approximations-based learning schemes for approachability in them.

The paper is organized as follows. In section 2, we state and prove Blackwell’s approachability theorem for MDPs. We also give a reinforcement learning algorithm for approachability when the transition kernels are not known. In section 3, we give Blackwell’s approachability theorem for Stackelberg stochastic games and give a learning algorithm. In section 4, we conclude with some observations and point out some interesting future directions.

2. A Blackwell’s Approachability Theorem for MDPs

2.1. Preliminaries We consider an MDP with a finite state space $\mathcal{S}$ and a finite action space $\mathcal{A}$. Let $p(\cdot, \cdot, \cdot)$ be the transition kernel that governs the system evolution such that $p(s, a, \cdot) \in \mathcal{P}(\mathcal{S})$ for all $(s, a) \in \mathcal{S} \times \mathcal{A}$. Let $\zeta$ be a vector-cost function, $\zeta : \mathcal{S} \times \mathcal{A} \to \mathbb{R}^K$. For a given state $s \in \mathcal{S}$ and any action $a \in \mathcal{A}$, $\zeta(s, a) = [c_1(s, a), \ldots, c_K(s, a)]^\top$ where $\zeta_j : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ for $1 \leq j \leq K$. We assume that the cost function is bounded and without loss of generality assume that $|c_j(s, a)| \leq 1$, $1 \leq j \leq K$, $\forall s \in \mathcal{S}$, $\forall a \in \mathcal{A}$.

For any arbitrary set $B$, let $\mathcal{P}(B)$ denote the space of probability measures over the set $B$. A stationary policy is a mapping $\pi : \mathcal{S} \to \mathcal{P}(\mathcal{A})$ such that for any state $s \in \mathcal{S}$ and an action $a \in \mathcal{A}$, $\pi(s, a)$ is the probability with which an action $a$ is chosen in state $s$ regardless of the previous history. Any stationary policy $\pi$ and an initial state $s_0 \in \mathcal{S}$ induces a Markov chain on $\mathcal{S}$. We make the following assumption:

**Assumption 2.1.** The Markov chain induced by any stationary randomized policy is irreducible.

This is a standard assumption made in reinforcement learning theory.

Let time be discrete and $(s_n, a_n)$ be the state-action pair at time $n$. The average vector-cost incurred till time $n$ is denoted by $x_n = \frac{1}{n} \sum_{m=1}^{n} \zeta(s_m, a_m)$. It is well known that under any stationary policy, the average vector cost $x_n$ converges to $\zeta(\pi)$ where

$$
\zeta(\pi) := \left[ \sum_{s,a} \eta^\pi(s) \, \pi(s,a) \, c_1(s,a), \ldots, \sum_{s,a} \eta^\pi(s) \, \pi(s,a) \, c_K(s,a) \right] \dagger,
$$

and $\eta^\pi$ is the stationary distribution of the Markov chain induced by the policy $\pi$.

Our objective is to specify the conditions under which a given closed set is approachable and prescribe a policy (not necessarily stationary) for approachability. The notion of an approachable set is made precise in the following definition.

Let $D \subset \mathbb{R}^K$ be any given set. Then for $x \notin D$ we define $\|x - D\| := \inf_{y \in D} \|x - y\|$. Let $\sigma$ be a possibly non-stationary policy and for any initial state $s \in \mathcal{S}$ and $\mu_s(\sigma)$ be the induced probability distribution on the sequence of vectors $\{x_n\}_{n \geq 0}$.

**Definition 2.1 (Approachable Set).** A set $D \subset \mathbb{R}^K$ is approachable if there exists a policy $\sigma$ (possibly non-stationary) such that $\|x_n - D\| \to 0$, $\mu_s(\sigma)$ almost surely for all initial states $s \in \mathcal{S}$.

A set is approachable if and only if its closure is approachable and hence without loss of generality we can consider only closed sets $D$. We first restrict our attention only to convex $D$. Extension to non-convex sets is discussed in Section 2.2.4. Thus, let $D$ be a closed convex set in $\mathbb{R}^K$. For any $x \in \mathbb{R}^K \setminus D$, let $P_D(x)$ denote the (unique) projection of $x$ onto $D$. Let $\lambda(x) = (x - P_D(x)) / \|x - P_D(x)\|$.

For each $x \in \mathbb{R}^K \setminus D$, we define a scalar-valued MDP with the stage cost $\tilde{c}(s, a; x) = \langle \zeta(s, a), \lambda(x) \rangle$. With respect to this scalar MDP parametrized by $x$, we define the following.

$$
c^*(x) = \min_{\pi} \langle \zeta(\pi), \lambda(x) \rangle \tag{2.2}
$$

$$
\Pi(x) = \arg \min_{\pi} \langle \zeta(\pi), \lambda(x) \rangle. \tag{2.3}
$$
Thus, $\Pi(x)$ is the set of stationary policies which minimizes the infinite horizon average cost

$$\lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} \tilde{c}(s_m, a_m; x)$$

and $c^*(x)$ is the corresponding optimal cost.

### 2.2. Approachability Theorem for MDPs

In this section, we state and prove our approachability theorem for MDPs.

#### 2.2.1. Sufficient Condition and Strategy for Approachability for Convex Sets

**Theorem 2.1.** (i) (Sufficient Condition) A closed convex set $D$ is approachable in an arbitrary MDP (satisfying Assumption 2.1) if the following holds: For every $x \in \mathbb{R}^K \setminus D$, there exists a (possibly non-unique) stationary randomized policy $\pi(x)$ such that $\langle \tilde{c}(\pi(x)) - P_D(x), \lambda(x) \rangle \leq 0$.

(ii) (Strategy for Approachability) A strategy for approachability is: at each time step $n + 1$, select the action $a_{n+1}$ according to a policy $\pi_{x_n}$ such that $\pi_{x_n} \in \Pi(x_n)$.

The geometric meaning of the above theorem is intuitive. For every point $x$ outside $D$ the player has a strategy $\pi_x$ such that the expected vector-cost corresponding to that policy lies in the halfspace containing $D$ defined by the supporting hyperplane for the set $D$ at $P_D(x)$. However, the strategy for approachability is slightly non-obvious. Since we are changing the policy at each time instant, it is not obvious how the corresponding MDP gets enough time to converge to the value corresponding to that policy. Later, we will show that this indeed happens via a stochastic approximation scheme.

Note that there is a modeling issue here. If one knows or can compute a policy $\bar{\pi}$ such that $c(\bar{\pi}) \in D$, all one has to do is to implement it, to reach $D$. Thus, the above model has a somewhat awkward viewpoint that while at time $n$, $\pi_{x_n}$ can be figured out, no such $\bar{\pi}$ can be. However, our purpose in analyzing this case is merely to pave way for the learning scheme that appears in the next section.

We first give an overview of the proof idea. The detailed proof is given in Section 2.2.2.

Average vector-cost till time step $n + 1$ can be written as

$$x_{n+1} = \frac{1}{n+1} \sum_{m=0}^{n+1} c(s_m, a_m) = x_n + \frac{1}{n+1} (\tilde{c}(s_{n+1}, a_{n+1}) - x_n)$$

$$= x_n + \frac{1}{n+1} (\tilde{c}(s_{n+1}, \pi_{x_n}) - x_n + M_{n+1}^{(1)})$$

(2.4)

where $\gamma(n) = 1/(n + 1)$, $c(s_{n+1}, \pi_{x_n}) = \sum_{a \in A} \pi_{x_n}(s_{n+1}, a) c(s_{n+1}, a) / \pi_{x_n}(s_{n+1}, a)$ and $M_{n+1}^{(1)} = c(s_{n+1}, a_{n+1}) - \tilde{c}(s_{n+1}, \pi_{x_n})$. The key idea in the analysis of (2.4) is to show that it asymptotically tracks the differential inclusion given by

$$\dot{x}(t) \in \tilde{d}(x(t)) - x(t).$$

(2.5)

where

$$\tilde{d}(x) := \{ \tilde{c}(\pi) : \pi \in \Pi(x) \}.$$ 

Then by showing that the dynamics given by (2.5) converges to the set $D$, we will conclude that the sequence $\{x_n\}$ also converges to the same set a.s.
2.2.2. Proof of Theorem 2.1  Before we give the detailed proof, it is necessary to verify the existence of a solution to the differential inclusion (2.5). We first state some regularity properties of (2.5).

**Proposition 2.1.** For each \( x \in \mathbb{R}^K \setminus D \),

(i) \( \sup_{y \in \mathcal{d}(x)} \|y\| < K(1 + \|x\|) \).

(ii) \( \mathcal{d}(x) \) is convex, compact and upper semicontinuous.

Both are rather straightforward to prove using standard analysis techniques and hence we omit the proof. It is well known that the differential inclusion in (2.5) with the regularity conditions given by Proposition 2.1 admits a solution through every initial point [3].

Even though equation (2.4) appears as a standard single timescale stochastic approximation iteration, it is much more complicated than that and involves an implicit multiple timescale process. The sequence \( \{x_n\} \) is 'affected' by the process \( \{s_n\} \) running in the background on the true or 'natural' timescale which corresponds to the time index ' \( n \) ' itself. Since \( \gamma_1(n) \) can be considered as time steps and \( \gamma_1(n) \to 0 \), the process \( \{x_n\} \) evolves on a slower timescale than the process \( \{s_n\} \).

The analysis involved is usually called 'averaging the natural timescale' and is described in detail in [11, Section 6.2]. Here we give only the results relevant for our problem. Readers are referred to [11, Section 6.2] for proofs and details.

[5], [6] contain pioneering work on stochastic approximations with differential inclusion. We shall, however, refer to [11] as a common source for all facts regarding stochastic approximations for convenience and uniformity of notation.

The basic approach to the analysis of (2.4) is to construct a suitable continuous interpolated trajectory \( \mathcal{I}(t), t \geq 0 \) and show that it asymptotically almost surely approaches the solution set of (2.5). This is done as follows: Define \( t(0) = 0, t(n) = \sum_{m=0}^{n-1} \gamma_1(m), m \geq 1 \). Clearly \( t(n) \uparrow \infty \). Let \( I_n := [t(n), t(n + 1)), n \geq 0 \). Define a continuous, piecewise linear \( \mathcal{I}(t), t \geq 0 \) by \( \mathcal{I}(t(n)) = x_n, n \geq 0 \), with linear interpolation in each interval \( I_n \). That is,

\[
\mathcal{I}(t) = x_n + (x_{n+1} - x_n) \frac{t - t(n)}{t(n+1) - t(n)}, \quad t \in I_n
\]

Define a \( \mathcal{P}(\mathcal{S} \times \mathcal{A}) \)-valued random process \( \mu(t) = \mu(t, (s, a)), t \geq 0 \), by

\[
\mu(t) := \delta_{(s_n, a_n)}, \quad t \in I_n, \quad n \geq 0,
\]

where \( \delta \) is the Kronecker delta function. Also, define for \( t > v \geq 0 \), and for \( B \) Borel in \([v, t] \),

\[
\mu_v(B \times (s, a)) := \frac{1}{t - v} \int_B \mu(y, (s, a))dy.
\]

Two necessary conditions for the analysis in [11] to carry through are:

1. Almost surely, \( \sup_n \|x_n\| < \infty \).

2. Almost surely, for any \( t > 0 \), the set \( \{\mu_v, v \geq 0\} \) remains tight.

Since we are dealing with only finite spaces, we don't need to worry about the measurability issues discussed in [11]. For the same reason, conditions 1 and 2 are also true. Let \( \Psi \) be the set of ergodic occupation measures over the set \( \mathcal{S} \times \mathcal{A} \), i.e., any \( \psi \in \Psi \) can be decomposed as \( \psi(s, a) = \eta^\pi(s) \pi(s, a) \) where \( \pi(\cdot, \cdot) \) is a stationary randomized policy and \( \eta^\pi(\cdot) \) is an invariant probability measure under the policy \( \pi \). For any \( x \in \mathbb{R}^K \setminus D \), let \( \Psi(x) \subset \Psi \) be such that any \( \psi \in \Psi(x) \) can be decomposed as \( \psi(s, a) = \eta^\pi_x(s) \pi_x(s, a) \) with \( \pi_x \in \Pi(x) \).

For \( \nu \in \mathcal{P}(\mathcal{S} \times \mathcal{A}) \), define

\[
\tilde{h}(x, \nu) := \sum_{(s, a) \in \mathcal{S} \times \mathcal{A}} (c(s, a) - x) \nu(s, a)
\]
For $\mu(\cdot)$ defined in (2.7), consider the non-autonomous o.d.e.

$$\dot{x}(t) = \tilde{h}(x(t), \mu(t))$$  \hspace{2cm} (2.8)

Let $x^*(t), t \geq v$, denote the solution to (2.8) with $x^*(v) = \pi(v)$, for $v \geq 0$.

A set $A \in \mathbb{R}^K$ is said to be an invariant set for the differential inclusion (2.5) if for $x(0) \in A$ there is some trajectory $x(t), t \in (-\infty, \infty)$, that lies entirely in $A$. An invariant set is said to be an internally chain transitive invariant set if for any $x, y \in A$ and any $\epsilon > 0, T > 0$, there exists an $n \geq 1$ and points $x_0 = x, x_1, \ldots, x_{n-1}, x_n = y$ in $A$, such that the trajectory of (2.5) initiated at $x_i$ meets with the $\epsilon$-neighborhood of $x_{i+1}$ for $0 \leq i < n$ after a time $t \geq T$.

We use the following important result from [11].

**Theorem 2.2.** [11, Theorem 7, Chapter 6]

Almost surely, $\{\pi(v + \cdot), v \geq 0\}$ converge to an internally chain transitive invariant set of the differential inclusion

$$\dot{x}(t) \in \{\tilde{h}(x, \nu) : \nu \in \Psi(x)\} := \mathcal{d}(x(t)) - x(t)$$  \hspace{2cm} (2.9)

as $t \uparrow \infty$. In particular, $\{x_n\}$ converge a.s. to such a set.

We now show that any path corresponding to the differential inclusion dynamics given by (2.5) converges to the set $D$. Some definitions are in order: A compact invariant set $M$ is called an attractor of a dynamical system if it has an open neighborhood $O$ such that every trajectory in $O$ remains in $O$ and converges to $M$. The largest such $O$ is called the domain of attraction. An attractor $M$ is called a global attractor if the domain of attraction is $\mathbb{R}^K$.

**Proposition 2.2.** The set $D$ is a global attractor for the differential inclusion specified by (2.5).

**Proof:** Consider the Lyapunov function $V(x) = \min_{z \in D} \frac{1}{2} \|x - z\|^2$. We first note that by Danskin’s Theorem ([7], p. 717), $\nabla V(x) = (x - P_D(x))$. So, $\frac{d}{dt} V(x(t)) = \langle \nabla V(x(t)), \dot{x}(t) \rangle = \langle x(t) - P_D(x(t)), y(t) \rangle$ for $y(t) \in \mathcal{d}(x(t)) - x(t)$.

By assumption, there exists a policy $\pi(x)$ such that $\langle \xi(\pi(x)), \lambda(x) \rangle \leq \langle P_D(x), \lambda(x) \rangle$. Then, for any optimal policy $\pi_x \in \Pi(x)$ which minimizes $\pi \mapsto \langle \xi(\pi), \lambda(x) \rangle$, we have $\langle \xi(\pi_x), \lambda(x) \rangle \leq \langle P_D(x), \lambda(x) \rangle$. So for any $x(t), \langle \xi(\pi_{x(t)}), x(t) - P_D(x(t)) \rangle \leq 0$ and hence $\langle \xi(\pi_{x(t)}), x(t) - P_D(x(t)) \rangle \leq -\|x(t) - P_D(x(t))\|^2$. This gives

$$\frac{d}{dt} V(x(t)) \leq -2V(x(t)) \implies V(x(t)) \leq V(x(0)) e^{-2t}.$$  

Thus, $D$ is a global attractor. \hfill \Box

Using Theorem 2.2 and Proposition 2.2, we now prove the Approachability Theorem for MDPs, i.e., Theorem 2.1.

**Proof:** (of Theorem 2.1)

By Theorem 2.2 $\{x_n\}$ converges a.s. to an internally chain transitive invariant set of the differential inclusion given by (2.5). Since $D$ is a global attractor, this internally chain transitive invariant set is a subset of $D$ [11]. Hence, $\{x_n\}$ converges a.s. to $D$. \hfill \Box

**2.2.3. Necessary Condition** We now show that the approachability condition given in Theorem 2.1 is also a necessary condition and thus give a complete characterization of the approachable convex sets in MDPs.

**Proposition 2.3 (Necessary Condition).** If a closed convex set $D$ is approachable in an arbitrary MDP (satisfying Assumption 2.1), then

(i) every half-space containing $D$ is approachable, and

(ii) for every $x \in \mathbb{R}^K \setminus D$, there exists a (possibly non-unique) stationary randomized policy $\pi(x)$ such that $\langle \xi(\pi(x)) - P_D(x), \lambda(x) \rangle \leq 0$.  

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Let \( x \in \mathbb{R}^K \setminus D \) and \( H_x \) be the supporting half-space to the set \( D \) at the point \( P_D(x) \) given by
\[
H_x := \{ y \in \mathbb{R}^K : \langle y - P_D(x), \lambda(x) \rangle \leq 0 \}.
\]
Since every half-space containing \( D \) is approachable, there exists a policy \( \sigma \) (possibly non-stationary) for the player such that \( \limsup_{n \to \infty} \langle x_n - P_D(x), \lambda(x) \rangle \leq 0 \). Since \( \langle x_n - P_D(x), \lambda(x) \rangle \) is bounded,
\[
\inf_{\lambda \in \Sigma} \mathbb{E}_\sigma \left[ \lim_{n \to \infty} \langle x_n, \lambda(x) \rangle \right] \leq \langle P_D(x), \lambda(x) \rangle
\]
The LHS of equation (2.10) is the optimal expected average cost corresponding to an MDP with scalar stage cost \( \tilde{c}(s, a; x) = \langle \mathcal{g}(s, a), \lambda(x) \rangle \). And from the theory of MDPs [9], there exists a stationary randomized policy \( \pi(x) \) such that this cost is equal to \( \langle \mathcal{g}(\pi(x)), \lambda(x) \rangle \). \( \square \)

### 2.2.4. Extension to Non-Convex Sets

Here we give the sufficient condition and strategy for approachability when the target set is non-convex. Note that, given an \( x \in \mathbb{R}^K \setminus D \), the closest point to \( x \) in \( D \) may not be unique. Let \( \tilde{P}_D(x) \) be the set of all such points.

**Theorem 2.3.**  
(i) (Sufficient Condition) A closed set \( D \) is approachable in an arbitrary MDP (satisfying Assumption 2.1) if the following holds: For every \( x \in \mathbb{R}^K \setminus D \) and for every \( P_D(x) \in \tilde{P}_D(x) \), there exists a (possibly non-unique) stationary randomized policy \( \pi(x) \) such that \( \langle \mathcal{g}(\pi(x)) - P_D(x), \lambda(x) \rangle \leq 0 \).

(ii) (Strategy for Approachability) A strategy for approachability is: at each time step \( n + 1 \), select a \( P_D(x) \in \tilde{P}_D(x) \), and select the action \( a_n \) according to a policy \( \pi_{x_n} \) such that \( \pi_{x_n} \in \Pi(x_n) \).

The condition given above is not necessary for approachability. This can be easily seen by considering a set \( D \) as the union of two disjoint convex sets.

The proof technique used in Theorem 2.1 is not directly applicable here because the Lyapunov function defined there may not be differentiable when the set \( D \) is non-convex. We overcome this difficulty by using *semidifferentials* and a general version of the *envelope theorem* [4].

We first give some some basic definitions and results from [4, Pages 29, 42-46] that is necessary for the proof.

**Definition 2.2 (Semi-differentials).** Let \( V : \mathbb{R}^K \to \mathbb{R} \). The super and sub-differential of \( V \) (or semi-differentials) at \( x \), \( D^+V(x) \) and \( D^-V(x) \), are defined as
\[
D^+V(x) := \left\{ p \in \mathbb{R}^K : \limsup_{y \to x, y \in \mathbb{R}^K} \frac{V(y) - V(x) - p \cdot (y - x)}{|x - y|} \leq 0 \right\}
\]
\[
D^-V(x) := \left\{ p \in \mathbb{R}^K : \liminf_{y \to x, y \in \mathbb{R}^K} \frac{V(y) - V(x) - p \cdot (y - x)}{|x - y|} \geq 0 \right\}
\]

Let \( V \) be such that
\[
V(x) := \inf_{z \in D} g(x, z)
\]
where \( g : \mathbb{R}^K \times \mathbb{R} \to \mathbb{R} \). Assume that,
\[
(A1) \ g \text{ is bounded and } g(\cdot, z) \text{ differentiable at } x \text{ uniformly in } z,
\]
\[
(A2) \ z \mapsto D_z g(x, z) \text{ is continuous and } z \mapsto g(x, z) \text{ lower semicontinuous}
\]
where \( D_z g \) is the partial derivative of \( g \) w.r.t. \( x \). Let
\[
\tilde{P}_D(x) := \arg\min_{z \in \mathbb{D}} g(x, z) := \{ z \in D : V(x) = g(x, z) \}, \quad Y(x) := \{ D_z g(x, z) : z \in \tilde{P}_D(x) \}
\]
Also, the (one-sided) directional derivative of \( V \) in the direction of \( q \) is
\[
D^+V(x)(q) := \lim_{h \to 0^+} \frac{V(x + hq) - V(x)}{h}.
\]
We use the following general version of the *envelope theorem* given below.
Proposition 2.4. [4, Proposition 2.13, Page 44] Let $D$ be a compact set and $g$ satisfies assumptions (2.11)-(2.12). Then,

\[
Y(x) \neq \emptyset \\
D^+ V(x) = \overline{co} Y(x) \\
D^- V(x) = \begin{cases} 
\{ y \} & \text{if } Y(x) = \{ y \} \\
\emptyset & \text{if } Y(x) \text{ is not a singleton} 
\end{cases}
\]

Moreover, $V$ has the (one-sided) directional derivative in any direction $q$, given by

\[
D^+ V(x)(q) = \min_{y \in V(x)} y \cdot q = \min_{p \in D^+ V(x)} p \cdot q
\]

Proof: Proof is similar to that of Theorem 2.1 except the fact that the Lyapunov function may not be differentiable when the set $D$ is non-convex. We overcome this difficulty by using semidifferentials.

Let $V(x) = \min_{z \in D} \frac{1}{2} \| x - z \|^2$ be the Lyapunov function. Then, $D^+ V(x) = \overline{co}\{(x - P_D(x)), P_D(x) \in \tilde{P}_D(x)\}$. Let $v(t) = V(x(t))$ and by chain rule [3, Proposition 7, Page 288],

\[
D^+ v(t) = D^+ V(x(t))(\dot{x}(t)) = D^+ V(x(t))(y(t)) \quad \text{for } y(t) \in \overline{d}(x(t)) - x(t).
\]

As before, we can also show that, for all $P_D(x(t)) \in \tilde{P}_D(x(t))$,

\[
\langle g(\pi_x(t)) - x(t), x(t) - P_D(x(t)) \rangle \leq -\| x(t) - P_D(x(t)) \|^2 = -2V(x(t)) = -2v(t).
\]

Then,

\[
D^+ v(t) = \min_{P_D(x(t)) \in \tilde{P}_D(x(t))} \langle g(\pi_x(t)) - x(t), x(t) - P_D(x(t)) \rangle \leq -2v(t)
\]

and by [3, Proposition 8, Page 289], we get

\[
v(t) - v(0) + 2 \int_0^t v(\tau) d\tau \leq 0.
\]

Now, by Gronwall’s inequality,

\[
v(t) \leq v(0) e^{-2t}.
\]

The rest of the proof is same as before. \hfill \Box

Remark 2.1. (i) We note that our approachability theorems are similar to that of Shimkin and Shwartz [16] but weaker than that of Milman [15] which does not require Assumption 2.1. However, this generality comes at the cost of an inability to develop a learning algorithm for approachability. Our approachability strategy is different from that in [16] (and also [15]), and this has necessitated a completely different proof from any of the prior works. Moreover, we are able to give a learning version of the approachability strategy, something quite difficult from either of the prior works. Kamal [13] gives a stochastic approximation-inspired iterative scheme for approachability but it is not a learning scheme as it depends on knowing the model fully. Thus, it seems that ours is the first approachability result for MDPs which uses a stochastic approximation scheme for its proof, and as a bio-product yields a natural learning scheme.

(ii) We also note that all these works address the approachability question for general stochastic games whereas our above theorem is for MDPs. We give an approachability theorem and a learning scheme for Stackelberg stochastic games in Section 3. In general, a learning scheme for a general stochastic games is a long-standing open problem. We are able to give a learning scheme for a special case, namely the Stackelberg stochastic game.
2.3. Reinforcement Learning Algorithm for Blackwell Approachability in MDPs

In this section, we introduce a reinforcement learning algorithm for approachability in multi-objective MDPs. The Approachability theorem for MDPs shows that if the agent selects her action at time step \( n + 1 \) according to the policy \( \pi_{x_n} \) such that \( \pi_{x_n} \in \Pi(x_n) \), then \( x_n \) approaches the desired set \( D \). Given \( x_n \), such a policy can be easily computed if one knows the transition kernel \( p(\cdot, \cdot, \cdot) \).

The problem of ‘learning’ arises when this transition kernel is unknown but one has access to a simulation device that can generate, for any \( s \in S, a \in A \), an independent \( S \)-valued random variable whose probability law is \( p(s, a, \cdot) \). So, the objective of a learning algorithm is to ‘learn’ such a policy \( \pi_{x_n} \) at each time step \( n \) using this simulation device. We first review some basic theory for average-cost MDPs from [1].

It is known that under Assumption 2.1, one can associate a value function with the problem of finding the optimal cost of a standard average-cost MDP, i.e, one can find \( V : S \rightarrow \mathbb{R} \) and a scalar \( c_{opt} \) such that they satisfy the dynamic programming equation

\[
V(s) = \min_a \left[ \tilde{c}(s, a; x) + \sum_{s'} p(s, a, s') V(s') \right] - c_{opt}, \quad s \in S
\]  

Also \( c_{opt} \) is the optimal cost which is unique. \( V(\cdot) \) is unique only up to an additive constant.

The same dynamic programming equation can be written in terms of the ‘Q-value’ defined by the expression in the square brackets on the right as

\[
Q(s, a) = \tilde{c}(s, a; x) + \sum_{s'} p(s, a, s') \min_b Q(s', b) - c_{opt}, \quad s \in S, a \in A
\]

with \( V(s) = \min_a Q(s, a) \). Equation (2.14) is useful because it can be shown that a stationary policy \( \pi \) is optimal if and only if \( \pi(s) \in \arg \min \min_b Q(s', b) \), \( \forall s \).

Typically, the optimal average cost (along with the corresponding optimal value function or the optimal Q function) is computed by using a technique called Relative Value Iteration (RVI),

\[
V_{n+1}(s) = \min_a \left[ \tilde{c}(s, a; x) + \sum_{s'} p(s, a, s') V_n(s') - f(V_n) \right], \quad s \in S
\]

where \( f \) is any Lipschitz function satisfying the following properties: For an all 1 vector \( e \), \( f(e) = 1 \), \( f(y + ce) = f(x) + c \) and \( f(cy) = cf(x) \) for \( c \in \mathbb{R} \). As a simple example, we can set \( f(V) = V(s_0) \) for a fixed \( s_0 \in S \). It is known that \( V_n \rightarrow V \) such that \( f(V) = c_{opt} \). Similarly, one can specify RVI for the Q function as

\[
Q_{n+1}(s, a) = \tilde{c}(s, a; x) + \sum_{s'} p(s, a, s') \min_b Q_n(s', b) - f(Q_n), \quad s \in S, a \in A
\]

We kept \( x \) fixed in the foregoing. We continue to do so in what follows and suppress the \( x \) dependence in order to simplify the notation. Both the RVI algorithms given above make use of the knowledge of the transition kernel \( p(\cdot, \cdot, \cdot) \). In [1], Abounadi, et al. proposed a learning algorithm for computing the optimal cost and optimal policy of an average-cost MDP. The algorithm is called RVI Q-learning.

We first assume that we have access to a simulation function \( \xi \) such that

\[
\xi : S \times A \times [0, 1] \rightarrow S
\]

and \( P(\xi(s, a, \omega) = s') = p(s, a, s') \) where \( P(\cdot) \) is taken with respect to the uniform random variable \( \omega \in [0, 1] \). The synchronous RVI Q-learning algorithm is given by

\[
Q_{n+1}(s, a) = Q_n(s, a) + \gamma(n) \left( \tilde{c}(s, a; x) + \min_b Q_n(\xi_n, b) - f(Q_n) - Q_n(s, a) \right)
\]
where \( \xi_n^s = \xi(s, a, \omega_n) \) and \( \omega_n \) is independent and uniformly distributed on \([0, 1]\). Here, \( \gamma(n) \) is a standard stochastic approximation step size satisfying the conditions

\[
\sum_n \gamma(n) = \infty, \quad \sum_n \gamma^2(n) < \infty. \tag{2.19}
\]

Similarly, the asynchronous version of RVI Q-learning is given by

\[
Q_{n+1}(s, a) = Q_n(s, a) + \gamma(n) \nu(s, a, n) \left( \bar{c}(s, a; x) + \min_b Q_n(\xi^s, b) - f(Q_n) - Q_n(s, a) \right) \tag{2.20}
\]

where \( \nu(s, a, n) := \sum_{m=0}^n I\{ (s, a) = (s_n, a_n) \} \) and \( \gamma(n) \) satisfy the additional conditions

\[
\sup_n \frac{\gamma(y[n])}{\gamma(n)} < \infty, \quad \sum_{m=0}^{\lfloor y[n] \rfloor} \gamma(m) \to 1 \text{ uniformly in } y \in (0, 1). \tag{2.21}
\]

Also, the relative sampling frequency of state-action pairs should be bounded away from zero, i.e.,

\[
\liminf_{n \to \infty} \frac{\nu(s, a, n)}{n+1} > 0 \text{ a.s. } \forall (s, a) \in S \times A. \tag{2.22}
\]

We can rewrite the synchronous RVI Q-learning equation (2.18) as

\[
Q_{n+1}(s, a) = Q_n(s, a) + \gamma(n) \left( T(Q_n)(s, a) - f(Q_n) - Q_n(s, a) + M_{n+1}^{(2)} \right) \tag{2.23}
\]

where the mapping \( T \) is defined as

\[
(TQ)(s, a) = \bar{c}(s, a; x) + \sum_{s'} q(s, a, s') \min_b Q(s', b), \tag{2.24}
\]

and

\[
M_{n+1}^{(2)} = \bar{c}(s, a; x) + \min_b Q_n(\xi_n, b) - (TQ_n)(s, a). \tag{2.25}
\]

The RVI Q-learning algorithm given in equation (2.23) is in the form of a standard stochastic approximation algorithm. The corresponding o.d.e. limit is

\[
\dot{Q}(t) = c(T'(Q(t)) - Q(t)) \tag{2.26}
\]

where \( T'(Q) = T(Q) - f(Q) \) and \( c > 0 \). Using standard stochastic approximation theory, it can be shown that the sequence \( Q_n \) asymptotically tracks this o.d.e. So, if we can show that this o.d.e. converges to \( Q^* \) such that \( f(Q^*) = c_{opt} \), then convergence of the RVI Q-learning algorithm to \( \alpha^* \) can be deduced from that. The following relevant result is available in [1].

**Theorem 2.4 ([1]).**  
(i) \( Q^* \) is the globally asymptotically stable equilibrium point for (2.26).  
(ii) The sequence \( \{Q_n\} \) given by the synchronous RVI Q-learning algorithm (2.18) or its asynchronous version (2.20) is bounded almost surely.  
(iii) In both the synchronous and the asynchronous RVI Q-learning algorithms, if \( \{Q_n\} \) remains bounded almost surely, then \( Q_n \to Q^* \).

Now we give our \( Q \)-learning algorithm for approachability in MDPs: Select action \( a_{n+1} \) according to the policy \( \pi_n \) and update \( x_n, Q_n, \pi_n \) as

\[
x_{n+1} = x_n + \gamma_1(n) (\bar{c}(s_{n+1}, a_{n+1}) - x_n), \quad \gamma_1(n) = 1/(n+1), \tag{2.27}
\]

\[
Q_{n+1}(s, a) = Q_n(s, a) + \gamma_2(\nu(s, a, n)) I\{ (s_n, a_n) = (s, a) \} \left( \bar{c}(s, a; x_n) + \min_b Q_n(s_{n+1}, b) - f(Q_n) - Q_n(s, a) \right), \tag{2.28}
\]

\[
\pi_{n+1}(s, \cdot) \in \arg \min_a Q_{n+1}(s, a), \tag{2.29}
\]
where $\gamma_2(n)$ satisfy the condition (2.19) - (2.21). It is clear that $\gamma_1(n)$ also satisfy these conditions. Moreover, $\gamma_2(n)$ should satisfy the condition

$$\frac{\gamma_1(n)}{\gamma_2(n)} \to 0 \quad (2.30)$$

The synchronous case is analogously written.

Remark 2.2. The proof of convergence of the above requires the condition given in (2.22). This is not ensured by the above choice of $\pi_{n+1}$, so one usually employs some randomization to ensure adequate ‘exploration’ implicit in the condition (2.22). One way to do so is to choose for some $0 < \epsilon < 1$, $\pi_{n+1}(s, \cdot) = (1 - \epsilon)\eta + \epsilon\kappa$, where $\eta(\arg \min(Q_{n+1}(s, \cdot))) = 1$ and $\kappa$ is uniform on $A$. This however ensures only near-optimality in online scenarios. This is standard for Q-learning and we shall not discuss it in detail.

Under the above assumptions we have the following theorem.

Theorem 2.5. Assume that the stochastic approximation step sizes $\gamma_1(n), \gamma_2(n)$ and the relative sampling frequency $\nu(s, a, n)$ satisfy the assumptions (2.19) - (2.22) and (2.30). Then, in both the synchronous and asynchronous reinforcement learning algorithms for Blackwell’s Approachability, we shall not discuss it in detail.

Proof: For notational simplicity, we consider only the synchronous case. First, we rewrite the equations (2.27) and (2.28) as

$$x_{n+1} = x_n + \gamma_1(n) \left( c(s_{n+1}, \pi_n) - x_n + M_{n+1}^{(1)} \right),$$

$$Q_{n+1}(s, a) = Q_n(s, a) + \gamma_2(n) \left( T(Q_n)(s, a) - f(Q_n) - Q_n + M_{n+1}^{(2)} \right),$$

where $c(s_{n+1}, \pi_n) = \sum_{a \in A} c(s_{n+1}, a)\pi(s, a)$, $M_{n+1}^{(1)} = c(s_{n+1}, a_{n+1}) - c(s_{n+1}, \pi_n)$ and $M_{n+1}^{(2)}$ is as given in (2.25). This is a standard two timescale stochastic approximation iteration where $x_n$ is on a slower timescale compared to $Q_n$. We first claim that $x_n$ can be considered ‘quasi-static’ for the analysis of (2.32). For this, we rewrite (2.31) as

$$x_{n+1} = x_n + \gamma_2(n) \left( \xi_n + M_{n+1}^{(3)} \right),$$

where $\xi_n = \frac{\gamma_1(n)}{\gamma_2(n)}(c(s_{n+1}, \pi_n) - x_n)$ and $M_{n+1}^{(3)} = \frac{\gamma_1(n)}{\gamma_2(n)}M_{n+1}^{(1)}$. Clearly, $\xi_n \to 0$ almost surely. Then, by [11, Section 2.2], $(x_n, Q_n)$ will converge to the internal chain transitive invariant set of the o.d.e.

$$\dot{x}(t) = 0, \dot{Q}(t) = T'(Q(t)) - Q(t).$$

Then, by Theorem 2.4, $Q_n - Q^*(x_n) \to 0$ a.s., where $Q^*(x_n)$ is the optimal $Q$ function of the scalar-valued MDP with cost function $c(s, a; x_n)$. Then, we can conclude that $(x_n, \pi_n)$ converges to the set $\{(x, \pi_x) : x \in \mathbb{R^K}, \pi_x \in \Pi(x)\}$. It follows that $c(s, \pi_n) - c(s, \pi_{x_n}) \to 0$ almost surely.

Now, we rewrite equation (2.31) as

$$x_{n+1} = x_n + \gamma_2(n) \left( (c(s_{n+1}, \pi_{x_n}) - x_n) + (c(s_n, \pi_n) - c(s_n, \pi_{x_n})) + M_{n+1}^{(1)} \right)$$

$$= x_n + \gamma_2(n) \left( (c(s_{n+1}, \pi_{x_n}) - x_n) + M_{n+1}^{(2)} \right),$$

(2.34)

where $\epsilon_n^{(2)} = (c(s_n, \pi_n) - c(s_n, \pi_{x_n}))$. From the argument above, it is clear that $\epsilon_n^{(2)} \to 0$ almost surely. Then, the asymptotic behavior of (2.34) is the same as that of the equation

$$x_{n+1} = x_n + \gamma_2(n) \left( (c(s_{n+1}, \pi_{x_n}) - x_n) + M_{n+1}^{(1)} \right),$$

(2.35)

because $\{\epsilon_n^{(2)}\}$ will contribute only an additional error term which is asymptotically negligible. (See, e.g., [11], p. 17.) This is the same as equation (2.4) and hence, by Theorem 2.1, $\|x_n - D\| \to 0$ almost surely. □
3. Blackwell’s Approachability Theorem for Stackelberg Stochastic Games  In this section, we now show analogous approachability theorems and learning algorithms for Stackelberg Stochastic games. We note that computation of equilibria of general stochastic games is a longstanding open problem. Computational algorithms and learning schemes are only known for some special cases, such as zero-sum and the single controller case. The results from the previous section are not immediately relevant here because there are two decision makers now with a general sum game between them.

3.1. Preliminaries  Consider a stochastic game with two players, finite state space $S$, finite action space $A = A^1 \times A^2$ where $A^i$ is the action of player $i$, $i = 1, 2$. An element $a = (a^1, a^2) \in A$ is called an action vector. Let $p(\cdot, \cdot) \in \mathcal{P}(S)$ be the transition kernel that governs the system evolution, $p(s, a, \cdot) \in \mathcal{P}(S)$ for all $s \in S, a \in A$. Let $c^i : S \times A \rightarrow \mathbb{R}^K$ be the vector-cost function for player $i$, and for a given state $s \in S$ and an action vector $a \in A$, $c^i(s, a) = [c^i_1(s, a), \ldots, c^i_K(s, a)]^\top$ where $c^i_j : S \times A \rightarrow \mathbb{R}$ for $1 \leq j \leq K$. We assume that the cost function is bounded and without loss of generality assume that $|c^i_j(s, a)| \leq 1, 1 \leq j \leq K, 1 \leq i \leq 2, \forall s \in S, \forall a \in A$. The average vector-cost incurred for player $i$ till time $n$ is denoted by $x^i_n = \frac{1}{n} \sum_{m=1}^{n} c^i(s, a)$. As we have argued in the introduction, developing a ‘natural and dynamic’ learning algorithm for a general stochastic game is inherently difficult. Here we consider a natural relaxation of the problem as ‘games against nature’, and formalize it as a Stackelberg stochastic game. In a Stackelberg stochastic game, at each time step $n$, player 1 takes an action $a^1_n$ first. Player 2 (adversary) observes this action and then selects her action $a^2_n$.

Let $\Sigma^i$ be the set of behavioral strategies of player $i$, $i = 1, 2$. A pair of strategies $(\sigma^1, \sigma^2) \in \Sigma^1 \times \Sigma^2$ together with an initial state $s$ induces probability distribution $\mu_s(\sigma^1, \sigma^2)$ on the sequence of vectors $\{x^i_n, i = 1, 2\}$. Our objective is to specify the conditions under which a given closed set is approachable and prescribe a strategy for approachability. The notion of an approachable set for stochastic games is made precise in the following definition.

**Definition 3.1 (Approachable Set).** A closed set $D$ is approachable for player $i$ if there exists a behavioral strategy $\sigma^i$ for player $i$ such that $\|x^i_n - D\| \rightarrow 0$, $\mu_s(\sigma^i, \sigma^{-i})$-almost surely for all $\sigma^{-i} \in \Sigma^{-i}$, and for all initial states $s \in S$.

**Remark 3.1.** In the following, we analyze the approachability conditions and the algorithm for player 1 (against the strategies of the adversary) and hence, we drop the superscript 1. Thus, $x_n$ indicates $x^1_n$, $\xi$ indicates $\xi^1$, etc.

Let $\Psi$ be the set of ergodic occupation measures over the set $S \times A$ such that any $\psi \in \Psi$ can be decomposed as $\psi(s, a^1, a^2) = \eta^\psi(s) \pi^1(\cdot|s) \pi^2(\cdot|s, a^1)$ where $\pi^1(\cdot|s) \in \mathcal{P}(A^1)$, $\pi^2(\cdot|s, a^1) \in \mathcal{P}(A^2)$, and $\eta^\psi$ is an invariant probability measure over $S$ induced by the policy pair $(\pi^1, \pi^2)$. For any $\psi \in \Psi$, define the ergodic vector-cost function for player $i$ as

$$
\xi^i(\psi) : \Psi \rightarrow \mathbb{R}^K = \left[ \sum_{s, a^1, a^2} \psi(s, a^1, a^2) c^i_1(s, a^1, a^2), \ldots, \sum_{s, a^1, a^2} \psi(s, a^1, a^2) c^i_K(s, a^1, a^2) \right]^\top
$$

(3.1)

Note that $\xi^i(\psi)$ is the expected average vector-cost for player 1 if both players play a pair of stationary strategies $(\pi^1, \pi^2)$ such that $\psi$ is the ergodic occupation measure induced by $(\pi^1, \pi^2)$. We make the following assumption.

**Assumption 3.1.** The Markov chain induced by any pair of stationary strategies $(\pi^1, \pi^2)$ by the players is irreducible.

3.2. Approachability Theorem for Stackelberg Stochastic Games  In this section, we state and prove our approachability theorem for Stackelberg stochastic games.
3.2.1. Sufficient Condition and Strategy for Approachability of Convex Sets As in the MDP case, we first restrict our attention only to a convex set \( D \). Extension to the non-convex set is given in Section 3.2.4.

We start by defining a few quantities. We reuse some of the notations from Section 2. Recall that for any \( x \in \mathbb{R}^K \setminus D \), let \( P_D(x) \) denote the (unique) projection of \( x \) onto \( D \). Let \( \lambda(x) = (x-P_D(x))/\|x-P_D(x)\| \). For every \( x \in \mathbb{R}^K \setminus D \), we define a scalar-valued Stackelberg stochastic game with the stage cost \( \tilde{c}(s,a;x) = \langle \tilde{c}(s,a), \lambda(x) \rangle \). With respect to this game, we define the following quantities:

\[
c^*(x) := \min_{\pi^1} \max_{\pi^2} \left\{ \langle \psi(\pi^1,\pi^2) \rangle : \psi(s,a^1,a^2) = \eta^\psi(s,\pi^2(a^2|s, a^1) \pi^1(a^1|s) \right\} \tag{3.2}
\]

\[
\Gamma_{\pi^1}(x) = \arg \max_{\pi^2} \left\{ \langle \psi(\pi^1,\pi^2) \rangle : \psi(s,a^1,a^2) = \eta^\psi(s,\pi^2(a^2|s, a^1) \pi^1(a^1|s) \right\} \tag{3.3}
\]

\[
\Pi^1(x) := \arg \min_{\pi^1} \max_{\pi^2} \left\{ \langle \psi(\pi^1,\pi^2) \rangle : \psi(s,a^1,a^2) = \eta^\psi(s,\pi^2(a^2|s, a^1) \pi^1(a^1|s) \right\} \tag{3.4}
\]

\[
\Psi(x) := \{ \psi \in \Psi : \psi(s,a^1,a^2) = \eta^\psi(s,\pi^2(a^2|s, a^1) \pi^1(a^1|s), \pi^1 \in \Pi^1(x), \pi^2 \in \Gamma_{\pi^1}(x) \} \tag{3.5}
\]

\[
\overline{\Psi}(x) = \{ \psi \in \Psi : \langle \psi(\lambda(x) \rangle \le c^*(x) \} \tag{3.6}
\]

**Theorem 3.1.** (i) (Sufficient Condition) A closed convex set \( D \) is approachable from all initial states in a Stackelberg stochastic game (satisfying Assumption 3.1) if for every \( x \in \mathbb{R}^K \setminus D \) there exists a (possibly non-unique) occupation measure \( \psi \), \( \psi(s,a^1,a^2) = \eta^\psi(s,\pi^2(a^2|s, a^1) \pi^1(a^1|s) \), such that \( \langle \psi \rangle = c^*(x) \) and \( \langle \psi - P_D(x), \lambda(x) \rangle \le 0 \).

(ii) (Strategy for Approachability) A strategy for approachability is: At every time instant \( n+1 \), select action \( a^1_{n+1} \) according to the policy \( \pi^1_{x_n} \) such that \( \pi^1_{x_n} \in \Pi^1(x_n) \).

### 3.2.2. Proof of Theorem 3.1

The proof here is also based on stochastic approximation ideas. However, since the system evolution now depends on actions taken by two players (who are not necessarily zero-sum adversaries), it is much more complicated. This requires additional work which we show below. Other details are similar to the MDP case, and are either summarized or omitted.

The average vector-cost till time step \( n+1 \) can be written as

\[
x_{n+1} = \frac{1}{n+1} \sum_{m=0}^{n+1} c(s_m,a_m) = x_n + \gamma_1(n) (\langle c(s_{n+1},a_{n+1}) - x_n \rangle \\
= x_n + \gamma_1(n) \left( (\langle c(s_{n+1},\pi_{x_n},a_{n+1}^2) - x_n \rangle + M_{n+1}^{(1)} \right) \tag{3.7}
\]

where \( \gamma(n) = 1/(n+1) \), \( c(s_{n+1},\pi_{x_n},a_{n+1}^2) = \sum_{a^1 \in A^1} \pi_{x_n}(a^1) c(s_{n+1},a^1,a_{n+1}^2) \) and \( M_{n+1}^{(1)} = \langle c(s_{n+1},a_{n+1}^1,a_{n+1}^2) - c(s_{n+1},\pi_{x_n},a_{n+1}^2) \rangle \). Similar to the MDP case, the key idea in the analysis of (3.7) is to show that it asymptotically tracks the differential inclusion dynamics given by

\[
\dot{x}(t) \in w(x(t)) - x(t). \tag{3.8}
\]

where

\[
w(x) := \overline{\mathcal{C}} \{ \tilde{c}(\tilde{\psi}) : \tilde{\psi} \in \overline{\Psi}(x) \}.
\]

Then, by showing that the dynamics given by (3.8) converges to the set \( D \), we will conclude that the sequence \( \{x_n\} \) also converges to the same set a.s.

We start with the following proposition.

**Proposition 3.1.** The sets \( \Psi(x) \) and \( \overline{\Psi}(x) \) are compact, non-empty and the set-valued maps \( \chi : x \mapsto \Psi(x) \), \( \tilde{\chi} : x \mapsto \overline{\Psi}(x) \) are upper semi-continuous.
Proof: Consider the Stackelberg stochastic game wherein at each time instant \( n \), the state is \( X_n \in S \) and player 1 moves first and generates a randomized action \( Z_n^1 \in A^1 \) according to conditional law \( \pi^1(\cdot | X_n) \), followed by player 2 who observes \( Z_n^1 \) and chooses a randomized action according to the conditional law \( \pi^2(\cdot | X_n, Z_n^1) \). Thus, player 2 faces an average reward MDP with state process \((X_n, Z_n^1, Z_n^2)\), \( n \geq 0 \), and transition probability \( \tilde{p}(\hat{s}, \hat{a}^1) (s, a^2) := p(s, a^1, a^2, s)\pi^1(\hat{a}^1|s) \). For a given \( x \in \mathbb{R}^k \setminus D \), the optimal adversarial policy \( \pi^2 \) by player 2 for the scalar-valued MDP with cost function \( \langle c(\cdot, a, z), \lambda(x) \rangle \) can be characterized as the maximizer in the associated dynamic programming equation, similar to equation (2.13) as

\[
V((s, a^1)) = \max_{\pi^2} \left( \sum_{(\hat{s}, \hat{a}^1)} \pi^2(a^2|s, a^1) \hat{p}((\hat{s}, \hat{a}^1)|(s, a^1), a^2) \left( \langle c(s, a^1, a^2), \lambda(x) \rangle + V((\hat{s}, \hat{a}^1)) \right) \right) - \beta.
\]

(3.9)

This has a solution \((V, \beta)\) where \( \beta \) is unique and \( V \) is unique up to an additive scalar and can be rendered unique by arbitrarily fixing, say \( V((s_0, a_0^1)) = 0 \). Then, \( \Gamma_{\pi^1}(x) \) defined above is simply the set of maximizers on the R.H.S. Being the set of maximizers of an affine function on a convex compact set, it is a non-empty convex compact set.

Now consider the MDP, with state process \( \{X_n\} \) and stationary randomized policies \( \pi((a^1, a^2)|s) := \pi^1(a^1|s)\pi^2(a^2|s, a^1) \) where \( \pi^1(\cdot|s) \in \mathcal{P}(A^1) \) and \( \pi^2(\cdot|s, a^1) \in \Gamma_{\pi^1}(x) \). That is, the action space for the problem is \( \Phi := \{\pi^1 \in \mathcal{P}(A^1) \times \Gamma_{\pi^1}(x) \} \) with the relative topology inherited from \( \mathcal{P}(A^1 \times A^2) \). By our definition of \( \Gamma_{\pi^1}(x) \), the optimal policy for this MDP corresponds to the min-max policy for the original problem, i.e., belongs to \( \Pi^1(x) \). This in turn is given by the minimizers on the right hand side of the dynamic programming equation

\[
\hat{V}(s) = \min_{\pi \in \Phi} \left( \sum_{\hat{s}} \sum_{a^1, a^2} \pi^1(a^1|s)\pi^2(a^2|s, a^1)p(s, a^1, a^2, \hat{s}) \left( \langle c(s, a^1, a^2), \lambda(x) \rangle + \hat{V}(\hat{s}) \right) \right) - \hat{\beta},
\]

(3.10)

which has a solution \((\hat{V}, \hat{\beta})\), where \( \hat{\beta} \) is unique and \( \hat{V} \) is unique up to an additive scalar and can be rendered unique by arbitrarily fixing, say \( \hat{V}(s_0) = 0 \). In either (3.9) or (3.10), if we replace \( x \) by \( x_n, x_n \to x_\infty \), then a subsequential limit \((V_{\infty}^n, \beta_{\infty}^n, (\hat{V}_{\infty}^n, \hat{\beta}_{\infty}^n))\) of the corresponding \((V^1 = V_{\infty}^n, \beta^1 = \beta_{\infty}^n, (\hat{V}^1 = \hat{V}_{\infty}^n, \hat{\beta}^1 = \hat{\beta}_{\infty}^n))\) must satisfy the respective dynamic programming equations (3.9),(3.10), with \( V_{\infty}^n((s_0, a_0^1)) = \hat{V}_{\infty}^n(s_0) = 0 \). By the uniqueness claim above, they are the appropriate value functions for \( x = x_\infty \). Furthermore, if we pick \( \pi_n^2 \) to be a maximizer on the right hand side of (3.9) for \((V_{\infty}^n, \beta_{\infty}^n)\), \( n \geq 1 \), any limit point thereof as \( n \uparrow \infty \) must be a maximizer of the same for \( n = \infty \). A similar argument works for the minimizers of (3.10). It is easy to deduce from this that the graph of \( \lambda \) is closed. Hence, \( \lambda \) is upper semi-continuous. This in particular implies, by our definition of \( \Psi(\cdot) \), that \( c^*(\cdot) \) is lower semi-continuous. The claim regarding \( \hat{\Psi}(\cdot) \) follows from this.

First we will prove some conditions that will ensure the well posedness of (3.8).

**Proposition 3.2.** For each \( x \in \mathbb{R}^k \setminus D \),

(i) \( \sup_{y \in \Xi(x)} \|y\| < K(1 + \|x\|) \).

(ii) \( w(x) \) is convex, compact and upper semicontinuous.

**Proof:** (i) is obvious from the boundedness assumption. For (ii), \( w(x) \) is convex by definition. Now, consider the mapping \( h(x) := \{c(\cdot) : \psi \in \hat{\Psi}(x)\} \). Since \( \hat{\Psi}(x) \) is compact, and \( c(\cdot) \) is continuous, \( h(x) \) is compact. Then, the closed convex hull of \( h(x) \), \( w(x) \), is also compact. The upper semicontinuity of \( h \) is clear from the upper semicontinuity of \( \hat{\psi} \). Now the closed convex hull of \( h(x) \), \( w(x) \), is also upper semicontinuous, by [11, Lemma 5, Chapter 5].

Now we prove Theorem 3.1.
Proof: The remaining details of the proof are very similar to that for MDPs. As before we construct the interpolated trajectory $\bar{x}(t), t \geq 0$ of (3.7) as defined in equation (2.6). Using the same arguments and then by applying the result of Theorem 2.2 we can show that almost surely, $\{\bar{x}(v+\cdot), v \geq 0\}$ converge to an internally chain transitive invariant set of the differential inclusion given by (3.8). In particular this implies that $\{x_n\}$ converge a.s. to such a set.

Consider the same Lyapunov function $V(x) = \min_{z \in D} \frac{1}{2} \|x - z\|^2$. Since $\nabla V(x) = (x - P_D(x))$, $\frac{d}{dt} V(x(t)) = (\nabla V(x(t)), \dot{x}(t)) = (x(t) - P_D(x(t)), y(t))$ for $y(t) \in w(x(t)) - x(t)$.

By Proposition 3.1 and our hypotheses, there exists an occupation measure $\psi$ such that $\psi(s, a^1, a^2) = \eta^\psi(s) \pi^2(a^2|s, a^1)\pi^1(a^1|s)$, and $c^*(x) = \langle \psi, \lambda(x) \rangle \leq \langle P_D(x), \lambda(x) \rangle$. Then for any policy $\tilde{\psi}_x \in \tilde{\Psi}(x)$ we have $\langle \psi_x, \lambda(x) \rangle \leq c^*(x) \leq \langle P_D(x), \lambda(x) \rangle$. So for any $x(t)$,

$$\langle \psi_{x(t)}, P_D(x(t)) \rangle, x(t) - P_D(x(t)) \leq 0 \text{ for all } \tilde{\psi}_{x(t)} \in \tilde{\Psi}(x(t))$$

and hence, $\langle \psi_{x(t)} - x(t), x(t) - P_D(x(t)) \rangle \leq -\|x(t) - P_D(x(t))\|^2$. This gives

$$\frac{d}{dt} V(x(t)) \leq -2V(x(t))$$

so that

$$V(x(t)) \leq V(x(0)) e^{-2t}$$

Thus, $D$ is a global attractor. Since $D$ is a global attractor, the internally chain invariant set corresponding to the differential inclusion (3.8) is a subset of $D$ [11]. Hence, $\{x_n\}$ converges to $D$.

3.2.3. Necessary Condition  We now give a necessary condition for approachability for convex sets.

Proposition 3.3 (Necessary Condition). If a closed convex set $D$ is approachable from all initial states in an arbitrary Stackelberg stochastic game (satisfying Assumption 3.1), then
(i) every half-space containing $D$ is approachable, and
(ii) there exists an occupation measure $\psi$, $\psi(s, a^1, a^2) = \eta^\psi(s) \pi^2(a^2|s, a^1)\pi^1(a^1|s)$, such that $\lambda(\psi) = c^*(x)$ and $\langle \psi - P_D(x), \lambda(x) \rangle \leq 0$.

Proof: Claim (i) is obvious. We now show that (i) implies (ii) and complete the argument.
Let $x \in \mathbb{R}^K \setminus D$ and $H_x$ be the supporting half-space to the set $D$ at the point $P_D(x)$ given by

$$H_x := \{y \in \mathbb{R}^K : \langle y - P_D(x), \lambda(x) \rangle \leq 0\}.$$ 

Since every half-space containing $D$ is approachable, there exists a strategy $\sigma^1$ for player 1 such that for any (Stackelberg) strategy $\sigma^2$ of player 2, $\limsup_{n \to \infty} \langle x_n - P_D(x), \lambda(x) \rangle \leq 0$, $\mu(\sigma^1, \sigma^2)$ a.s. Since $|\langle x_n - P_D(x), \lambda(x) \rangle|$ is bounded,

$$\inf_{\sigma^1} \sup_{\sigma^2} \mathbb{E}_{\mu(\sigma^1, \sigma^2)} \left[ \limsup_{n \to \infty} \langle x_n, \lambda(x) \rangle \right] \leq \langle P_D(x), \lambda(x) \rangle$$

(3.11)

Note that the L.H.S. of equation (3.11) is the min-max cost for player 1 in the average cost scalar-valued Stackelberg stochastic game with stage cost $\tilde{c}(s, a; x) = \langle \tilde{c}(s, a), \lambda(x) \rangle$. Then by the arguments in the proof of Proposition 3.1, there exists a $\psi \in \Psi(x)$ such that the L.H.S. is equal to $\langle \psi, \lambda(x) \rangle$. This completes the proof. □
3.2.4. Extension to Non-Convex Sets

We now give the approachability result when the target set is non-convex. The proof is the same as that of Theorem 3.1 except the fact that the Lyapunov function may be non-differentiable. We overcome this difficulty by considering the semidifferentials as we did for MDPs (in Theorem 2.3). Since the techniques are the same, we omit the proof.

As before, let \( \tilde{P}_D(x) \) be the set of points in \( D \) that are closest to \( x \in \mathbb{R}^K \setminus D \).

**Theorem 3.2.** (i) (Sufficient Condition) A closed convex set \( D \) is approachable from all initial states in the stochastic game satisfying Assumption 3.1 if for every \( x \in \mathbb{R}^K \setminus D \) and for each \( P_D(x) \in \tilde{P}_D(x) \), there exists an occupation measure \( \psi \), \( \psi(s,a^1,a^2) = \eta(s)\pi^2(a^2|s,a^1)\pi^1(a^1|s) \), such that such that \( \hat{c}(\psi) = c^*(x) \) and \( \langle \hat{c}(\psi) - P_D(x), \lambda(x) \rangle \leq 0 \).

(ii) (Strategy for Approachability) A strategy for approachability is: At every time instant \( n+1 \), select \( a_n = \pi^1_{x_n} \) such that \( \pi^1_{x_n} \in \Pi^1(x_n) \).

**Remark 3.2.** (i) Our approachability theorem is for a special case, namely the Stackelberg stochastic games while that of [15, 16, 13] is for general stochastic games. However, we are able to give a different approachable strategy based on stochastic approximations. This approachability strategy naturally leads to a dynamic learning algorithm.

(ii) The only learning algorithm for approachability in stochastic games we are aware of, is given in [14]. This algorithm is based on the approachable strategy in [16], and seems complicated and computationally impractical. Our learning algorithm is simpler but is applicable only to Stackelberg stochastic games. This is an interesting step in developing stochastic approximation based learning algorithms for general stochastic games, similar to reinforcement learning algorithms for MDPs. Such a goal, as we noted earlier is a longstanding open problem, and has been pursued without success for a long time.

3.3. A Learning Algorithm for Approachability in Stochastic Stackelberg Games

Approachability theorem for Stackelberg stochastic games that we proved before shows that if player 1 selects her action at time step \( n+1 \) according to the policy \( \pi^1_{x_n} \) then \( x_n \) approaches the desired set \( D \). So the objective of the algorithm is to ‘learn’ such a policy \( \pi_{x_n} \) at each time step \( n \). We give an algorithm which indeed does this.

We do this by considering the problem as learning in two coupled MDPs. Thus, most of the tools that we used in Section 2 can be applied directly. We give the asynchronous learning algorithm below. The synchronous scheme may be written analogously.

\[
x_{n+1} = x_n + \gamma_1(n) (\tilde{Q}_{n+1}(s,a^1,a^2))^{-1} (x_n), \quad \gamma_1(n) = \frac{1}{(n+1)},
\]

\[
\tilde{Q}_{n+1}(s,a^1,a^2) = \hat{\tilde{Q}}_n((s,a^1),a^2) + \gamma_2(n,s,a^1,a^2) I\{((s,a^1),a^2) = ((s,a^1),a^2)\}
\]

\[
\frac{c(s,a^1,a^2;x_n)}{\max Z Q_n((s_{n+1},a_{n+1}),z) - f(Q_n) - \tilde{Q}_n((s,a^1),a^2),}
\]

\[
\pi^1_{n+1}(s,a^1) = \arg \max_{a^2} \tilde{Q}_{n+1}(s,a^1,a^2),
\]

\[
\tilde{Q}_{n+1}(s,a^1) = \tilde{Q}_n((s,a^1),a^1) + \gamma_3(n,s,a^1,a^1) I\{s_n = s, a_n = a^1\}
\]

\[
\frac{c(s,a^1,a^2;x_n) + \min Y \tilde{Q}_n(s_{n+1},y) - f(Q_n) - \tilde{Q}_n(s,a^1),}
\]

\[
\pi^1_{n+1}(s,a) = \arg \min_{a^1} \tilde{Q}_{n+1}(s,a^1),
\]

where \( \gamma_1(n), \gamma_2(n), \gamma_3(n) \) satisfy the conditions in equations (2.19), (2.21), (2.30), with the additional stipulations that \( \gamma_3(n) = o(\gamma_2(n)), \gamma_3(n) = o(\gamma_1(n)) \). Also, \( \hat{\nu}(n,s,a^1,a^2) = \sum_{m=1}^n I\{(s,a^1,a^2) = (s_m,a^1,a^2_m), \hat{\nu}(n,s,a^1) = \sum_{m=1}^n I\{(s,a^1) = (s_m,a^1_m)\} \).

**Remark 3.3.** Here also, as observed in Remark 2.2, we need to randomize \( \pi^1_{n+1} \). We skip the details.
Theorem 3.3. The learning algorithm for approachability in Stackelberg stochastic game given by equation \((3.12) - (3.16)\), \(\|x_n - D\| \to 0\) almost surely.

Proof: Proof is very similar to that of Theorem 2.5 and hence we give only a sketch of the arguments. Here, \(\hat{Q}_n\) is the \(Q\)-function for the MDP faced by player 2 and note that \(\hat{Q}_n\) and \(\hat{\pi}^2_n\) are computed by player 1 only. \(\hat{Q}_n\) is the \(Q\)-function for the MDP faced by player 1.

Both \(x_n\) and \(\pi^1_n\) are on a slower timescale compared to \(\hat{Q}_n\) and can be considered ‘quasi-static’ for the analysis of \((3.13)\). Thus, by the results of \([1]\), we can conclude that \(\|\hat{\pi}^2_n - \Gamma_{\pi^1_n}(x_n)\| \to 0\) a.s.

Consider \((3.15)\) next. Again, \(x_n\) can be treated as quasi-static for this and \(\hat{Q}_n\), which is on a faster timescale, as quasi-equilibrated (i.e., \(\|\hat{\pi}^2_n - \Gamma_{\pi^1_n}(x_n)\| \to 0\) a.s.), whence

\[
\min_{a^1} \hat{Q}_n(s_n, a^1) - \min_{a^2} \max_{a^1} \hat{Q}_n(s_n, a^1, a^2) \to 0, \text{ a.s.}
\]

Then, one can conclude that \(\|\pi^1_n - \Pi^1(x_n)\| \to 0\) a.s. This implies that asymptotically player 1’s strategy is the same as the approachable strategy specified by Theorem 3.1 and hence by the same theorem \(\|x_n - D\| \to 0\).

These arguments can be made rigorous as in the proof of Theorem 2.5 by essentially following the steps in \([11]\). Since that is routine, we conclude the proof here. \(\square\)

4. Conclusion We have presented a simple and computationally tractable strategy for Blackwell’s approachability in MDPs and Stackelberg stochastic games. We have also given a reinforcement learning based algorithm to learn the approachable strategy when the transition kernels are unknown. The motivation for this came from multi-objective optimization and decision making problems in a dynamically changing environment. We note that while approachability questions are typically asked in the context of games, repeated or stochastic, they are basically about settings when decision-makers are faced with multiple objectives. The conditions for approachability for MDPs and stochastic games are very similar. However, in our stochastic approximations-based approach, the proof techniques for the Stackelberg stochastic game setting is much more complicated since we must account for system dynamics depending on actions taken by both players. The learning algorithms we devise for the two settings are also different since for MDPs we have a stochastic approximation scheme that ‘averages over the natural time scale’ while for the game setting, we have a multiple time scale stochastic approximation scheme.

There are many interesting related questions that one can possibly address in the future. Extension to MDPs and stochastic games with discounted reward is one problem but the solution is possibly very messy due to dependence on the initial state. The optimal rate of convergence for the approachability problems in such systems is another important problem that has never been addressed before, and likely very challenging.

References


