Communication Games on the Generalized Gaussian Relay Channel

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Abstract—Relaying is often seen as one of the means of increasing capacity of wireless networks [?]. Thus, various cooperative communication schemes for relaying have been proposed that achieve close to the information-theoretic network capacity. The premise behind such schemes is cooperative behavior of users. This, however, cannot be taken for granted. Users may be selfish and care only about their own rates. They might even strategically deviate from their agreed role in such cooperative communication schemes leading to a possible degradation for all. In this paper, we look at the generalized Gaussian relay channel model with two selfish users. A capacity-achieving communication scheme for such channel models was proposed by Sendonaris, et al. [?]. We show that under certain channel conditions, operating on a part of the Pareto-optimal boundary of the achievable rate region with this scheme is a Nash equilibrium, and in fact these are the only Nash equilibria. We establish the results for the one-way Gaussian relay channel, and then extend it to the two-way Gaussian relay channel model. These results may be seen as being in the same spirit as the recent results of Berry and Tse for the Gaussian interference channel [?], [?].

Index Terms—Cooperative Communications, Spectrum Sharing, Game Theory, Generalized Gaussian Relay Channel.

I. INTRODUCTION

The scarcity of spectrum is an impediment to the growth of more capable wireless networks. Several measures are sought to address this problem: Freeing up unused spectrum, sharing of spectrum through new paradigms such as cognitive radio sensing, as well as sophisticated communication schemes that rely on user cooperation. Such schemes involve not only on cooperative sharing of spectrum but also relaying. However, given that cooperation can lead to rate degradation, this may fail to happen with selfish users. This leads us to ask: Why would users cooperate in relaying? [?]. What happens if they don’t? [?]. We investigate these questions for a prototypical information-theoretic channel model that depends on such cooperation for relaying.

The (standard) relay channel model was introduced by van der Meulen [?]. This involved communication from a transmitter (node 1) to a receiver (node 3) with the aid of a passive relay (node 2), i.e., the relay node does not have any information of its own to send. An upper bound on the capacity of this channel was obtained by Cover and El Gamal [?]. Carleial introduced a more general version of the simple relay channel usually called the multi-access channel with generalized feedback (MAC-GF) [?], and which we will refer to as the generalized Gaussian relay (GGR) channel model. A more general encoding/decoding strategy for the MAC-GF channel was developed by Willems, et al. [?], [?]. They introduced backward decoding and used superposition block Markov decoding to derive an achievable rate region for the GGR channel model. In [?], Sendonaris, et al. looked at the channel model from a different perspective as a method of providing diversity to the system. In [?], it was shown that the GGR channel model can be considered as a practical model for achieving such cooperative diversity.

Recently, Cioffi, Tse and their co-workers have re-investigated the achievable rate region of the Gaussian interference channel with selfish users [?], [?]. Here, only transmission powers were regarded as the strategies of the players. A more information-theoretic formulation of the game on the interference channel was given in [?] where they also defined the Nash equilibrium region in the achievable rate region. They gave a complete characterization of such a region for the deterministic interference channel model [?]. This was extended to the symmetric two-user Gaussian interference channel model, and shown that when the almost capacity-achieving Han-Kobayashi scheme [?] is used, there are always Nash equilibria on the Pareto-optimal boundary of the achievable rate region [?]. The converse, however, has not been established (i.e., that there are no
other Nash equilibria). In [1], a problem with similar motivation has been investigated. However, the channel model investigated is a Gaussian interference channel with a passive relay whose achievable rate region was established in [2].

In this paper, we investigate the achievable rate region of the generalized Gaussian relay (or MAC-GF) channel with selfish users when the Sendonaris, et al. cooperative communication scheme is used by users. Users’ strategies include transmission powers and encoding rates, as well as a decision to not cooperate in accordance with the Sendonaris et al. scheme. We show that under certain channel conditions, there are Nash equilibrium rates on the Pareto-optimal achievable rate region boundary. And, in fact, these are the only Nash equilibria.

II. MODEL AND RELATED WORK

The generalized Gaussian relay channel model is shown in Figure ???. There are two users, both want to transmit data to a common destination. There are two differences from the standard relay channel model [2]. User $i$ has its own information to send, denoted by $W_i$. Both user can cooperate by relaying each others’ data in some fashion. This would increase both their transmission rates. (In the standard relay channel, user 2 may help user 1 to increase its data rate, and not the other way round). When users are not cooperating in their transmission, this model reduces to the standard two-user MAC channel for which the capacity region is well known.

Sendonaris, et al. [2] gave a communication scheme, encoding strategies that the transmitting users can use, and a decoding strategy to be used by the receiver. We refer the reader to [2] for more details. In this paper, we assume that either the two transmitting users do not cooperate at all, and if they do, the two transmitters and the receiver use the Sendonaris, et al. encoding/decoding scheme. However, since users are assumed selfish, even while “cooperating” they can deviate from the “optimal” Sendonaris scheme by using a different power, transmission rates, or can refuse to cooperate entirely.

The discrete time mathematical model of the channel in Figure ?? is given by

$$
\begin{align*}
Y_0 &= K_{10}X_1 + K_{20}X_2 + Z_0 \\
Y_1 &= K_{12}X_1 + Z_1 \\
Y_2 &= K_{21}X_2 + Z_2.
\end{align*}
$$

(1)

The $Y_0, Y_1, Y_2$ are the received signal at the destination 0, user 1 and user 2 respectively. $X_i$ is the signal transmitted by user $i$, and $Z_i$ is additive channel noise which is i.i.d. with distribution $\mathcal{N}(0,1)$. We assume that the transmission is done for $B$ blocks of length $n$, and both $B$ and $n$ are large. Both users employ a “cooperative” strategy based on the superposition block Markov encoding and backward decoding [2], [2]. More specifically, user 1 divides information $W_1$ into two parts: $W_{10}$ to be sent directly to the destination, and $W_{12}$ to be sent to the destination via user 2. In addition to these, it also sends an additional “cooperative” information to the destination for enhanced decoding. In any block, user 1’s transmitted signal $X_1$ is structured as

$$
X_1 = X_{10} + X_{12} + U_1
$$

(2)

where

$$
\begin{align*}
X_{10} &= \sqrt{P_{10}}X_{10}(W_{10}(i), W_{12}(i-1), W_{21}(i-1)) \\
X_{12} &= \sqrt{P_{12}}X_{12}(W_{12}(i), W_{12}(i-1), W_{21}(i-1)) \\
U_1 &= \sqrt{P_{U1}}U(W_{12}(i-1), W_{21}(i-1))
\end{align*}
$$

(3)

and the total power $P_1$ is divided into $(P_{10}, P_{12}, P_{U1})$ such that $P_1 = P_{10} + P_{12} + P_{U1}$. The user selects $X_{10}, X_{12}, U$ from i.i.d. samples from the $\mathcal{N}(0,1)$ distribution.

Here, $U_1$ denotes that part of the signal that carries the “cooperative” information. Power used for transmitting $X_{10}$ directly to the destination is denoted by $P_{10}$, which encodes $W_{10}$ at rate $R_{10}$. Power used for transmitting $X_{12}$ to user 2 is denoted $P_{12}$, which encodes $W_{12}$ to user 2 at the rate $R_{12}$, and power used to transmit $U_1$ to the destination is denoted $P_{U1}$, which encodes the “cooperative” information. User 2 constructs its transmit signal $X_2$ similarly.

Now, user 2 can construct its signal $X_2$ only if it can perfectly decode $W_{12}$. Thus, transmission rate $R_{12}$ (and power $P_{12}$) should be selected by user 1 so that $W_{12}$ can be decoded perfectly by user 2. Similarly, rate $R_{21}$ (and power $P_{21}$) should be selected by user 2 so that $W_{21}$ can be decoded perfectly by user 1.

It was shown in [2] that an achievable rate region with this scheme for the generalized Gaussian relay channel is given as the convex hull of all rate pairs $(R_1, R_2)$ such
that
\[ R_i = R_{i0} + R_{ij}, j \neq i, \]
with
\[ R_{12} \leq C \left( \frac{K_0^2 P_{12}}{1 + K_{12} P_{10}} \right) \]
\[ R_{21} \leq C \left( \frac{K_0^2 P_{21}}{1 + K_{21} P_{20}} \right) \]
\[ R_{10} \leq C \left( K_{01}^2 P_{10} \right) \]
\[ R_{20} \leq C \left( K_{02}^2 P_{20} \right) \]
\[ R_{10} + R_{20} \leq C \left( K_{10}^2 P_{10} + K_{20}^2 P_{20} \right) \]
\[ R_{10} + R_{12} + R_{20} + R_{21} \leq R_{\text{sum}} = C \left( K_{10}^2 P_{10} + K_{20}^2 P_{20} + 2K_{10} K_{20} \sqrt{P_{10} P_{20}} \right) \]

where \( C(x) := \frac{1}{2} \log(1 + x) \), for some power allocation satisfying \( P_{10} + P_{12} + P_{11} = P_1 \) and \( P_{20} + P_{21} + P_{22} = P_2 \).

The proof can be found in [?] and the references therein. The essential ideas are backward decoding and coherent combining of the cooperative signals \( U_1 \) and \( U_2 \). By virtue of the backward decoding, the destination will decode \( W_{12}(i) \) and \( W_{21}(i) \) from block \( i + 1 \). Thus, decoding the \( i \)th block involves decoding the messages \( W_{10}(i), W_{20}(i), W_{12}(i-1) \) and \( W_{21}(i-1) \). The receiver decodes \( W_{12} \) and \( W_{21} \) jointly which are encoded in the “cooperative” parts of the transmitted signals, \( U_1 \) and \( U_2 \). So, effectively the receiver has to decode three different streams of data, the joint stream \( (W_{12}, W_{21}) \), \( W_{10} \) and \( W_{20} \) from the aggregate signal. This is done in the same way as successive interference cancellation decoding in a 3-user MAC channel. As in the MAC channel, the receiver can decode these streams in any order. For example, the receiver can do joint decoding of \( W_{12}, W_{21} \) first, \( W_{10} \) second and \( W_{20} \) last. The achievable rate region is computed by considering all such possible decoding orders.

Both users are rational and selfish with their utility functions equal to the total rates \( R_1 \) and \( R_2 \) that they achieve. Hence, they will optimize their power allocation to maximize their individual rate.

A strategy of user \( i \) is his encoding rates \( (R_{i0}, R_{ij}) \) and power allocation \( (P_{i0}, P_{ij}, P_{ui}) \) such that \( P_{i0} + P_{ij} + P_{ui} \leq P_i, j \neq i \). Let \( s_i \) denote the strategy vector \( (R_{i0}, R_{ij}, P_{i0}, P_{ij}, P_{ui}) \). Each user receives a payoff equal to his achieved rate. Thus, \( U_i(s_1, s_2) = R_i = R_{i0} + R_{ij} \) if the power allocation for the two users satisfies their power budgets and their encoding rates satisfy the inequalities (??), else it is 0. Now, we define Nash equilibrium region for the generalized relay channel.

**Definition 1** (Nash Equilibrium region). A strategy pair \((s_1^*, s_2^*)\) is a Nash Equilibrium (NE), if for each \( i = 1, 2 \),
\[ U_i(s_1^*, s_2^*) \geq U_i(s_1, s_2^*), \forall s_1, j \neq i. \]

The Nash equilibrium region is the set of all Nash equilibrium encoding rates that are achievable at a Nash Equilibrium.

As usual, it says that no player can increase his utility rate by a unilateral deviation. The role of the receiver : As already mentioned, the receiver can do decoding in various orders. To determine if a particular rate pair \((R_1, R_2)\) is a NE, we need to know the actual decoding scheme/order being used by the receiver (among those that can be used to achieve that rate pair). The role of the receiver in the game is to a priori choose a decoding order, possibly after it knows the channel conditions \((K_{10}, K_{20}, K_{12}, K_{21})\). There are six possible decoding orders.

1. **Decoding order \( D_1 \)**: Receiver decodes \((W_{12}, W_{21})\) first, \( W_{10} \) next, and \( W_{20} \) last.
2. **Decoding order \( D_2 \)**: Receiver decodes \((W_{12}, W_{21})\) first, \( W_{20} \) next, and \( W_{10} \) last.
3. **Decoding order \( D_3 \)**: Receiver decodes \( W_{10} \) first, \((W_{12}, W_{21})\) next, and \( W_{20} \) last.
4. **Decoding order \( D_4 \)**: Receiver decodes \( W_{20} \) first, \((W_{12}, W_{21})\) next, and \( W_{10} \) last.
5. **Decoding order \( D_5 \)**: Receiver decodes \( W_{20} \) first, \( W_{10} \) next, and \((W_{12}, W_{21})\) last.
6. **Decoding order \( D_6 \)**: Receiver decodes \( W_{10} \) first, \( W_{20} \) next, and \((W_{12}, W_{21})\) last.

The achievable rate region corresponding to decoding order \( D_1 \) will be denoted by \( C(D_1) \).

**III. THE ONE-WAY RELAY CHANNEL**

In this section, we consider a special case of the generalized Gaussian relay channel model. We assume that user 2 can relay user 1’s data, but not the other way around. This condition can be included in the model easily by setting \( K_{21} = 0 \). This is done as a first step to solve the two-way relay channel problem but is also interesting in its own right.

Since \( K_{21} = 0 \), user 2 cannot relay any of its data through user 1 and hence \( R_{21} = 0 \). Since \( R_{21} \) is zero, user 2 won’t allocate any power for transmitting its data to user 1 and hence \( P_{21} = 0 \). As for the generalized relay channel, the achievable rate region for this one-way relay channel is given by the closure of the convex hull of all rate pairs \((R_1, R_2)\) such that \( R_1 = R_{10} + R_{12} \) and \( R_2 = R_{20} \), and \((R_{10}, R_{12}, R_{20})\) satisfy the achievable rate bounds (??) with \( K_{21} = 0 \) and \( R_{21} = 0 \), for some power allocation satisfying power budgets \( P_{10} + P_{12} + \)
$P_{U1} = P_1$ and $P_{U2} = P_2$. From the encoding scheme model in the equations (??) and bounds (??), we can see that for decoding order $D_1$, the maximum possible rates are given by

$$R_{12} \leq \tilde{R}_{12} := C\left(\frac{K_{12}^2(P_{12} + P_{U1}) + K_{20}^2P_{U2} + 2K_{10}K_{20}\sqrt{P_{U1}P_{U2}}}{K_{10}^2P_{10} + K_{20}^2P_{20}}\right)$$

$$R_{12} \leq \tilde{R}_{12} := C\left(\frac{K_{12}^2P_{12}}{1 + K_{12}^2P_{10}}\right)$$

$$R_{10} = C\left(\frac{K_{10}^2P_{10}}{1 + K_{20}P_{20}}\right)$$

$$R_{20} = C\left(\frac{K_{20}^2P_{20}}{1 + K_{20}P_{20}}\right)$$

$$R_{10} + R_{12} + R_{20} \leq R_{\text{sum}}.$$

### A. Equilibrium Analysis

We will proceed by proving a series of results. The main question is what rate pairs $(R_1, R_2)$ corresponds to a NE. We answer this by first characterizing NE rate pairs. The following proposition will give a restriction on the rate pairs $(R_1, R_2)$ that can be achieved at a Nash Equilibrium [?].

**Proposition 1.** If the rate pair $(R_1, R_2)$ is a Nash equilibrium, then $R_i \geq R_{i,\text{min}} := C\left(\frac{K_{i0}^2P_i}{1 + K_{i0}P_i}\right)$ for $i = 1, 2$.

**Proof:** Regardless of user $j$’s encoding scheme, user $i$ can always achieve a minimum rate $R_{i,\text{min}} = C\left(\frac{K_{i0}^2P_i}{1 + K_{i0}P_i}\right)$, by treating the signal from user $j$ as noise. Thus, if user $i$ gets a rate $R_i$ strictly less than $R_{i,\text{min}}$ by the cooperation scheme, he has an incentive to deviate from that cooperation scheme and get a higher rate. Thus, if $(R_1, R_2)$ is a NE, then $R_i \geq C\left(\frac{K_{i0}^2P_i}{1 + K_{i0}P_i}\right)$.

**Proposition 2.** Suppose the receiver uses the decoding scheme $D_1$, and user 2 transmits at the rate $R_2$ and his power allocation is $(P_{20}, P_{U2})$ such that $P_{20} + P_{U2} = P_2$.

(i) When $K_{12} < \frac{K_{10}}{\sqrt{1 + K_{20}P_{20}}}$, the best response of user 1 is the power allocation $P_{12} = P_{U1} = 0$, $P_{10} = P_1$.

(ii) When $\frac{K_{10}}{\sqrt{1 + K_{20}P_{20}}} \leq K_{12} < \sqrt{\frac{(K_{10}^2P_1 + K_{20}^2P_{U2})}{(1 + K_{20}P_{20})}}$, the best response of user 1 is the power allocation $P_{10} = P_{U1} = 0$, $P_{12} = P_1$.

(iii) When $K_{12} \geq \sqrt{\frac{(K_{10}^2P_1 + K_{20}^2P_{U2})}{(1 + K_{20}P_{20})}}$, the best response of user 1 is the power allocation $(P_{10}, P_{U1})$ such that $P_{10} + P_{12} + P_{U1} = P_1$ and he maximizes his rate $R_1$, i.e.,

$$\max_{P_{10}, P_{12}} R_{10} + R_{12}$$

s.t. $R_{12} \leq \{\tilde{R}_{12}, \tilde{R}_{12}\}$.

Thus, the feasible set can be divided into two regions: Region 1 corresponds to the power allocations such that $\tilde{R}_{12} \leq \tilde{R}_{12}$ and Region 2 corresponds to the power allocations such that $\tilde{R}_{12} \geq \tilde{R}_{12}$. We will now show that any power allocation in Region 1 will be sub-optimal unless it also belongs to Region 2. We can then restrict our attention only to Region 2 in which case $R_{12} = \tilde{R}_{12}$ in the optimization problem.

Consider any power allocation $(P_{10}, P_{12}, P_{U1})$, $P_{10} + P_{12} + P_{U1} = P_1$ such that $R_{12} < \tilde{R}_{12}$ and $R_{12} = \tilde{R}_{12}$.  Consider $P_{10}$ fixed, then we can decrease $P_{12}$ to $P_2 - \epsilon$ and increase $P_{U1}$ to $P_{U1} + \epsilon$ in such a way that the new power allocation is still in Region 1. This would increase $R_{12}$ while $R_{10}$ remains the same. This power reallocation can be done until $\tilde{R}_{12} = \tilde{R}_{12}$. Thus, the optimal power allocation in Region 1 is such that it is also in Region 2. Thus, in the maximization problem above we need to consider Region 2 only. In Region 2, $R_{12} = \tilde{R}_{12}$,
so substituting in the objective function, we have
\[
\max_{P_{10}, P_{12}} C \left( \frac{K^2_{10}P_{10}}{1 + K^2_{20}P_{20}} \right) + C \left( \frac{K^2_{12}P_{12}}{1 + K^2_{21}P_{10}} \right).
\]
Since \(C(\cdot)\) is a monotonically increasing function, after some simple algebra and using the fact that \(P_{12} + P_{10} = P_1 - P_{U1}\), this can further be reduced to
\[
\max_{P_{10}, P_{U1}} \frac{(1 + K^2_{20}P_{20} + K^2_{10}P_{10}) (1 + K^2_{12}(P_1 - P_{U1}))}{(1 + K^2_{20}P_{20}) (1 + K^2_{12}P_{10})}.
\]
For any given value of \(P_{U1}\), the optimization problem reduces to
\[
\max_{P_{10}} \frac{1 + K^2_{20}P_{20} + K^2_{10}P_{10}}{1 + K^2_{12}P_{10}}.
\]

The slope of the objective function is \(\frac{(K^2_{10} - K^2_{12}) (1 + K^2_{20}P_{20})}{(1 + K^2_{20}P_{20})^2}\). Clearly, the sign of the slope doesn’t depend on the optimization variable \(P_{10}\). This means that when \(K^2_{10} > K^2_{12}(1 + K^2_{20}P_{20})\), the slope is positive and hence the objective function is monotonically increasing. When \(K^2_{10} \leq K^2_{12}(1 + K^2_{20}P_{20})\), the slope is negative and the objective function is monotonically decreasing.

When \(K^2_{10} > K^2_{12}(1 + K^2_{20}P_{20})\) or equivalently when \(K_{12} < \frac{K^2_{10}}{\sqrt{(1 + K^2_{20}P_{20})}}\), user 1 is maximizing a monotonically increasing function. Thus, the solution for the optimization problem is the maximum possible value of \(P_{10}\). Since the optimization is for a fixed \(P_{U1}\), the maximum possible values of \(P_{10}\) is \(P_{10} = P_1 - P_{U1}\) and thus \(P_{12} = 0\). Now, our problem reduces to finding the optimum value \((P_{10}, P_{U1})\) which maximizes the rate \(R_1\). Since \(P_{12} = 0\), \(R_{12} = 0\), and user 1 is not relaying any data through user 2. Since there is no relaying, user 1 can set \(P_{10} = 0\) which makes \(P_{10} = P_1\) and maximizes his rate \(R_1 = R_{10}\). Thus the best response for user 1 under this channel condition is \(P_{12} = P_{U1} = 0\), \(P_{10} = P_1\).

When \(K^2_{10} \leq K^2_{12}(1 + K^2_{20}P_{20})\) user 1 maximizes a monotonically decreasing function. So, the obvious solution for user 1’s optimization problem is \(P_{10} = 0\). Then, \(P_{12} + P_{U1} = P_1\). Also, since \(P_{10} = 0\) we have \(R_{10} = 0\) and \(R_1 = R_{12}\). Now, our problem reduces to finding the values of \(P_{12}\) and \(P_{U1}\) such that it maximizes \(R_{12}\). When \(P_{10} = 0\), we can write \(R_{12}\) and \(R_{12}\) as a function of \(P_{12}\) as
\[
R_{12}(P_{U1}) = C \left( K^2_{12}(P_1 - P_{U1}) \right),
\]
\[
R_{12}(P_{U1}) = C \left( \frac{K^2_{10}P_{10} + K^2_{20}P_{20} + 2K_{10}K_{20}\sqrt{P_{10}P_{20}}}{1 + K^2_{20}P_{20}} \right).
\]
\(R_{12}(P_{U1})\) is a decreasing function of \(P_{U1}\) while \(R_{12}(P_{U1})\) is an increasing function of \(P_{U1}\). It is easy to infer the solution geometrically as shown in Figure 3.

When \(\hat{R}_{12}(0) < \tilde{R}_{12}(0)\) or equivalently when \(K_{12} < \sqrt{(K^2_{10}P_{10} + K^2_{20}P_{20}) / (1 + K^2_{20}P_{20})}\) it is clear that these curves won’t intersect and \(\tilde{R}_{12}(P_{U1}) < \hat{R}_{12}(P_{U1})\) for all values of \(P_{U1}\). Thus, the optimum power allocation \(P_{U1}\) for user 1 is the one that maximizes \(\tilde{R}_{12}(P_{U1})\), which obviously implies \(P_{U1} = 0\) (and thus \(P_{12} = P_1\)). Thus, the best response of user 1 under this channel condition is \(P_{10} = 0, P_{12} = P_1\).

When \(\hat{R}_{12}(0) \geq \tilde{R}_{12}(0)\) or equivalently when \(K_{12} \geq \sqrt{(K^2_{10}P_{10} + K^2_{20}P_{20}) / (1 + K^2_{20}P_{20})}\) these curves will intersect at a particular value \(P_{U1}\). Since both curves are non-negative and \(\hat{R}_{12}(P_1) = 0\), the intersection point \(P_{U1}\) will be less than \(P_1\). So, it is clear that the condition \(\hat{R}_{12}(P_{U1}) = \tilde{R}_{12}(P_{U1})\), which is a quadratic equation in terms of \(P_{U1}\), has a valid solution that maximizes the rate of user 1. At this power allocation, the rate of user 1 is given by \(R_1 = R_{12} = R_{12}(P_{U1}) = \tilde{R}_{12}(P_{U1})\). Thus, the best response of user 1 is \(P_{10} = 0\) and \(P_{U1} + P_{12} = P_1\) where \(P_{U1}\) is obtained as the solution of a quadratic equation.

Now, we characterize the best response of user 1 when \(K_{12} \geq K_{10}\).

**Lemma 1.** Suppose the decoding order is \(D_1\), and user 2’s rate is \(R_2\) and the power allocation is such that \(P_{20} + P_{U2} = P_2\). When \(K_{12} \geq K_{10}\) and \(R_2 \geq R_{2_{\min}}\), then the best response of user 1 is \(P_{10} = 0\) and \((P_{U1}, P_{12})\) such that \(P_{U1} + P_{12} = P_1\) where \(P_{U1}\) is obtained as the
unique non-negative solution of the quadratic equation \( \hat{R}_{12}(P_{U1}) = \hat{R}_{12}(P_{U1}) \). The resulting rate pair \((R_1, R_2)\) is on the Pareto-optimal boundary of the region \( C(D_1) \).

**Proof:** Clearly, \( K_{12} \geq K_{10} \) implies \( K_{12} \geq \frac{K_{10}}{\sqrt{1+K_{20}P_{20}}} \). So, by Proposition 2, the best response is \( P_{10} = 0 \). Since \( R_2 = R_{20} = C(K_{20}P_{20}) \geq R_{2,\text{min}} = C\left(\frac{K_{20}^2P_{20}}{1+K_{20}^2P_{20}}\right) \), we have, \( P_{20} \geq \frac{P_0}{1+K_{10}P_1} \).

Thus, \( \hat{R}_{12}(0) = C\left(\frac{1+K_{10}^2P_1+K_{20}^2P_2}{1+K_{20}^2P_{20}}\right) \leq C(K_{10}P_1) \leq C(K_{12}^2P_1) = \hat{R}_{12}(0) \). But the condition \( \hat{R}_{12}(0) \leq \hat{R}_{12}(0) \) implies that \( K_{12} \geq \sqrt{\left(\frac{K_{20}^2P_1+K_{20}^2P_2}{1+K_{20}^2P_{20}}\right)} \). Then by Proposition 2, \( P_{U1} \) is obtained as the solution the quadratic equation \( \hat{R}_{12}(P_{U1}) = \hat{R}_{12}(P_{U1}) \) and the best response for user 1 is \( P_{10} = 0 \) and \( P_{U1} + P_{12} = P_1 \). At this power allocation, as shown in the proof of Proposition 2, \( \hat{R}_{12} = \hat{R}_{12} \). Thus the rate of user 1 will be \( R_1 = R_{12} = R_{12}(P_{U2}) \). Then \( R_1 + R_2 = R_{12} + R_{20} = R_{\text{sum}} \). Thus the sum rate bound is tight and hence \((R_1, R_2)\) is on the Pareto-optimal boundary of \( C(D_1) \).

Now we state a similar result for user 2. The proof is similar to that of Lemma-1.

**Proposition 3.** Suppose the decoding order is \( D_1 \), \( K_{12} \geq K_{10} \), user 1’s rate \( R_1 \geq R_{1,\text{min}} \) and the power allocation is \( P_{10} = 0 \) and \( P_{U1} + P_{12} = P_1 \). Then the best response of user 2 is \( P_{20} + P_{U2} = P_2 \) and \( P_{U2} \) is obtained as the solution of a quadratic equation. The resulting rate pair \((R_1, R_2)\) is on the Pareto-optimal boundary of the region \( C(D_1) \).

Now, we want to know which rate pairs \((R_1, R_2)\) satisfying \( R_1 \geq R_{1,\text{min}} \) are a Nash Equilibrium.

**Theorem 1.** Suppose the channel conditions are such that \( K_{12} \geq K_{10} \). A rate pair \((R_1, R_2)\), satisfying \( R_i \geq R_{i,\text{min}} \) is a Nash equilibrium if and only if it is on the Pareto-optimal boundary of the region \( C(D_1) \).

**Proof:** Let \((R_1, R_2)\) be a Nash Equilibrium. Then the rate \( R_1 \) and the corresponding power allocation \((P_{10}, P_{12}, P_{U1})\) of user 1 will be the best response of user 1 against the rate \( R_2 \) and his power allocation \((P_{20}, P_{21}, P_{U2})\). The same is true for user 2 also. Then by Lemma-1 and Proposition 2, the rate pair obtained by these best response strategies will be on the Pareto-optimal boundary of the region \( C(D_1) \). Thus \((R_1, R_2)\) is on the Pareto-optimal boundary of the region \( C(D_1) \).

We now prove the converse. Let \((R_1, R_2)\) be on the Pareto-optimal boundary of the region \( C(D_1) \). As shown in the proof of Proposition 1, the power allocation \((P_{20}, P_{U2})\) for user 2 to achieve this rate \( R_2 \) is determined uniquely. Now, by Lemma 1, user 1’s achieved rate \( R_1 \) is obtained by the best response power allocation \((P_{10}, P_{12}, P_{U1})\) against the rate \( R_2 \) and the corresponding power allocation \((P_{20}, P_{U2})\) of user 2. Since, \( R_1 \) is obtained by the best response action, user 1 won’t deviate from the operating point \((R_1, R_2)\). Now, we consider user 2. He can deviate from the operating point \((R_1, R_2)\) and increase his rate only by increasing the power \( P_{20} \) and thus by decreasing \( P_{U2} \). However, when user 2 increases \( P_{20} \) and decreases \( P_{U2} \), the maximum rate \( \hat{R}_{12} \) that user 1 can get also decreases. However, since user 1, unaware of the deviation by user 2, is still transmitting at the earlier rate \( R_1 = \hat{R}_{12} = \hat{R}_{12} \). The actual transmission rate of user 1 will be more than the maximum possible data rate. Thus, the whole block is decoded erroneously, and so the receiver will not be able to decode the data stream from user 1 successfully. Since the decoding of user 1’s signal has failed, the receiver won’t be able to subtract away user 1’s signal from the received signal. Now, when the receiver tries to decode user 2, the whole power from user 1 will appear as noise and thus user 2 can only get an effective rate \( R'_2 = C\left(\frac{K_{20}^2P_{20}}{1+K_{20}^2P_{20}}\right) \). Clearly \( R'_2 \leq R_{2,\text{min}} \leq R_2 \). Thus, user 2 will not unilaterally deviate from the operating point \((R_1, R_2)\). Hence the rate pair \((R_1, R_2)\) on the Pareto-optimal boundary of the region \( C(D_1) \) corresponds to a NE.

So far we have considered a particular decoding order \( D_1 \). The achievable capacity region \( C(D_1) \) by this decoding order can be a subset of the capacity region obtained by some other decoding order. However, we can show that there is no loss of generality by considering decoding order \( D_1 \) when \( K_{12} \geq K_{10} \).

**Theorem 2.** Suppose the channel conditions are such that \( K_{12} \geq K_{10} \), then any rate pair \((R_1, R_2)\) in the achievable rate region can be achieved with decoding order \( D_1 \).

**Proof:** We prove this by considering all other decoding orders and comparing the achievable region with that of \( D_1 \).

In Lemma 1, under the assumption that \( K_{12} \geq K_{10} \), we have showed that \( P_{10} = 0 \) and hence \( R_{10} = 0 \). Thus the \( W_{10} \) stream is not present in the transmission. Thus, when \( K_{12} \geq K_{10} \), the decoding order \( D_1 \) is \((W_{12}, W_{20})\).

(1) Decoding Order \( D_2 \), \((W_{12}, W_{20}, W_{10})\): Under this decoding order, \( \hat{R}_{12} \) and \( \hat{R}_{12} \) will be the same as in \( D_1 \) (defined by \((?!)\)), and \( R_{20} = C\left(\frac{K_{20}^2P_{20}}{1+K_{20}^2P_{20}}\right) \) and \( R_{10} = C\left(\frac{K_{10}^2P_{10}}{1+K_{10}^2P_{10}}\right) \).

As in \( D_1 \), for any given power allocation for user 2 with \( P_2 = P_{20} + P_{U2} \), user 1’s best response is given by
the optimal power allocation of user 1 is $K_{10}$. When $K_{12} \geq K_{10}$, the objective function is monotonically decreasing in $P_{10}$. Thus the solution for the optimization problem is $P_{10} = 0$. Thus, $R_{10} = 0$. Now, the decoding order becomes $(W_{12}, W_{20})$ which is the same as in $D_1$ which implies $C(D_2) = C(D_1)$. In a similar way we can prove that $C(D_3) = C(D_1)$.

(2) Decoding Order $D_4$, $(W_{20}, W_{12}, W_{10})$: Since $K_{12} \geq K_{10}$, as in the decoding order $D_1$, we can show that the optimal power allocation of user 1 is $P_{10} = 0$ and $P_1 = P_{12} + P_{U1}$. So, as in decoding order $D_1$, $R_{10} = 0$ and $R_1 = R_{12} = C(K_{10}^2P_{12})$. Thus to get the same rate $R_1$, the power allocation of user 1, $(P_{12}, P_{U1})$, is the same in $D_1$ and $D_3$. Since $W_{20}$ is decoded first, receiver has to treat all the powers $P_1$ and $P_{12}$ as interference. Then, the rate $R_{20} = C \left( \frac{K_{10}^2P_{12}}{1 + K_{10}^2P_{20}} \right)$. But these interference terms are absent in $D_1$. Thus, to get the same rate $R_2$, $P_{20}$ should be more in $D_3$ compared to that in $D_1$. And hence the power $P_{U2}$ should be less compared to that in $D_1$. Since $P_{U1}$ remains the same, the sum rate $R_{sum}$ should be less in $D_3$ compared to $D_1$. Thus, $C(D_3)$ is worse than $C(D_1)$. As an illustration, with decoding order $D_3$, user 2 can get his maximum rate $R_{2,max} = C(K_{20}^2P_2)$ only when user 1 is not transmitting. So, in $D_3$, whenever user 2 is getting a rate $R_{2,max}$, user 1’s rate $R_1$ is zero. Thus, with this decoding order, the rate pair $(R_{1,min}, R_{2,max})$ is not achievable, whereas with decoding order $D_1$, it is achievable. In a similar way we can show that $C(D_6) = C(D_5)$ and both are strictly worse than $C(D_1)$.

Thus we have established that the achievable region $C(D_i)$ for $i = 2, \ldots, 6$ is same or worse than $C(D_1)$. Thus when $K_{12} \geq K_{10}$, $C(D_1)$ is the same as the achievable rate region of obtained by considering all the possible decoding order.

Now, this yields the following conclusion:

**Corollary 1.** When $K_{12} \geq K_{10}$, the rate pairs $(R_1, R_2)$, satisfying $R_i \geq R_{i,min}$ is a NE if and only if it is on the Pareto-optimal boundary of the achievable region of the one-way relay channel.

### IV. The Two-way Relay Channel

We now consider the two-way GGR channel. We show that the analysis and the results for one-way relay channel can be readily extended to the two-way relay channel. Again, we focus on the decoding order $D_1$.

The rate constraints on $R_{12}$ and $R_{21}$ are the same as in (3) along with a $R_{12} + R_{21} \leq C \left( \frac{K_{10}^2(P_{12} + P_{U1}) + K_{20}^2P_{U2} + 2K_{10}K_{20}\sqrt{P_{U1}P_{U2}}}{K_{10}^2P_{10} + K_{20}^2P_{20}} \right)$.

We consider three different channel conditions.

**Case 1**: $K_{12} \geq K_{10}$ and $K_{21} \geq K_{20}$

As in the one-way relay channel model, for a given power allocation of user 2, user 1 will maximize his rate $R_1 = R_{10} + R_{12}$. Here, expressions for $R_{10}$ and $R_{12}$ are all the same as in the one-way relay channel. Thus, the optimization problem is the same as before, and so is the solution. In one-way relay channel, we concluded that when $K_{12} \geq K_{10}$ user 1’s rate $R_1$ is maximized by the power allocation $P_{10} = 0$ and $P_{U1} + P_{12} = P_1$. Thus in two way relay channel also, when $K_{12} \geq K_{10}$, user 1’s best response is the same power allocation. Now, by symmetry, the same results is also true for user 2. Thus in this case the equilibrium strategies are such that $P_{10} = 0$, $P_{U1} + P_{12} = P_1$ and $P_{U2} = P_2$ which implies $R_1 = R_{12}$ and $R_2 = R_{21}$.

Suppose that user 2’s rate $R_2$ is fixed. Since $R_2$ is a monotone function of $P_{21}$ alone, given $R_{21}$ we can find $P_{21}$ uniquely. Then $P_{U2} = P_2 - P_{21}$ is also determined uniquely. Now for this particular rate $R_2$, $\hat{R}_{12}$ and $\hat{R}_{11}$ are given by

$$\hat{R}_{12}(P_{U1}) = C \left( \frac{K_{10}^2(P_1 - P_{21})}{1 + K_{10}^2P_{10}} \right)$$

and $\hat{R}_{11}(P_{U1}) = C \left( \frac{K_{10}^2P_1 + K_{20}^2P_{U2} + 2K_{10}K_{20}\sqrt{P_{U1}P_{U2}}}{1 + K_{20}^2P_{21}} \right) - R_2$.

When $R_{2} \geq R_{2,min}$, as in the one-way relay channel, the optimal value of $P_{U1}$ can be found by equating $\hat{R}_{12}(P_{U1})$ and $\hat{R}_{12}(P_{U1})$ and solving the resulting quadratic equation. As discussed in Lemma 1 and Proposition 2, the solution is unique. At this solution, the sum rate bound is tight and the the resulting rate pair $(R_1, R_2)$ will be on the Pareto-optimal boundary of $\mathcal{C}(D_1)$. By symmetry, we can do the similar analysis for user 2. Then by Theorem 2 we can conclude:

**Theorem 3.** Suppose the channel conditions are such that $K_{12} \geq K_{10}$ and $K_{21} \geq K_{20}$. Then a rate...
pair \((R_1, R_2)\), satisfying \(R_i \geq R_{i, \text{min}}\) is in the Nash Equilibrium region of the two-way relay channel if and only if it is on the Pareto-optimal boundary of the region \(C(D_1)\).

**Case 2 :** \(K_{12} \geq K_{10}\) and \(K_{21} < K_{20}\)

Since \(K_{12} \geq K_{10}\), as in the previous discussion, we can conclude that the optimal power allocation for user 1 is \(P_{10} = 0\) and \(P_{12} + P_{U1} = P_1\). Then the condition \(K_{21}^2(1 + K_{10}^2 P_{10}) < K_{20}^2\) becomes \(K_{21} < K_{20}\) and hence optimum power allocation for user 2 will be \(P_{12} = 0\) and \(P_{20} + P_{U2} = P_2\). This is exactly the one-way relay channel solution. Thus, under this channel condition, the two-way relay channel reduces to a one-way relay channel and all the results of the one-way relay channel holds.

**Case 3 :** \(K_{12} < K_{10}\) and \(K_{21} \geq K_{20}\)

This is the same as the Case 2, except the fact that the role of user 1 and user 2 are interchanged.

The results from the one-way relay channel are not directly applicable when the channel conditions are such that \(K_{12} < K_{10}\) and \(K_{21} < K_{20}\). The exact characterization of the Nash equilibrium region of the two-way relay channel under this channel conditions will be done in future work.

**V. Discussion and Further Work**

We have established that under channel conditions \(K_{12} \geq K_{10}\) and \(K_{21} \geq K_{20}\), all rate pairs \((R_1, R_2)\) such that \(R_i \geq R_{i, \text{min}}, i = 1, 2\) which are on the Pareto-optimal boundary of the achievable region of the generalized Gaussian relay channel correspond to a NE of the communication game between the users of the channel. Our preliminary results indicate that a similar conclusion may not hold for other channel conditions. This implies that there may be channel conditions under which we cannot take for granted that selfish, rational users would readily cooperate in a cooperative communication scheme for relaying. In future work, we hope to elaborate the exact channel conditions and communication schemes employed at the strategic equilibria when “cooperation” for relaying fails. We will also generalize the results to more than two users, and when the utility function of users may be any increasing concave function of their rate.

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**References**