Assortment Planning in School Choice  
[Preliminary Draft]

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In many public school systems across the US, school choice has become the preferred alternative to the traditional method of assigning each student to a neighborhood school. In a typical school choice system, each student submits a preference ranking for a set of schools called the student’s menu. The school board then assigns students to schools using the Gale-Shapley deferred acceptance (DA) algorithm, which takes into account priorities that differentiate certain types of students, as well as quotas for students at each school. The menus, priorities and quotas are policy levers with which the school board may induce socially desirable outcomes. This paper presents a systematic approach for optimizing these policy levers.

Our methodology is based on a novel connection between stable matching and assortment planning, which allows us to approximate school choice as a convex optimization problem. The key to solving this convex optimization is to find an assortment of schools for each student type that maximizes the sum of utilities for this type and externalities for others. We refer to this as the ”socially-optimal assortment planning” problem, and show that it is a generalization of the revenue-maximizing assortment planning problem studied in the revenue management literature. We give efficient algorithms for the socially-optimal assortment planning problem for the multinomial logit (MNL), nested logit, and Markov chain based utility distributions.

We demonstrate the effectiveness of our methodology by applying it to actual data from Boston Public Schools. We construct optimized menus and priority distributions that outperform the status quo, improving the expected utilities of students and the predictability of the assignment outcome while maintaining the same amount of busing.

1. Introduction

In many public school systems across the US, school choice has become the preferred alternative to the traditional method of assigning each student to a designated school based on home location. In a typical school choice system, each student is given a set of school options, which we refer to as the student’s menu, and submits a preference ranking of the schools in this menu. These preference rankings are collected by the school board many months before the school year starts, and the school board computes the school assignment using a centralized algorithm, which takes into account possible priorities between various types of students and admission quotas at schools.
In Boston, New York City, Chicago, Denver, New Orleans, Washington DC, among other cities, the assignment algorithm is the student-proposing deferred acceptance algorithm, originally proposed by Gale and Shapley (1962) and applied to school choice by Abdulkadiroğlu and Sönmez (2003). The algorithm computes a stable matching, which means that no student is rejected by a school that has left over seats, and that no student is rejected by a school that accepted another with a lower priority. Moreover, the algorithm is strategyproof, meaning that students have no incentives to misreport their preference rankings. This incentive property was especially important in the adoption of this algorithm in New York and Boston (see Abdulkadiroğlu et al. (2006) and Abdulkadiroğlu et al. (2009)).

In the student-proposing deferred acceptance algorithm (which we simply refer to as the DA algorithm), there are three important policy levers. The first is the menu of each student, which is the set of schools each student can rank in the preference submission. We assume that the menus are only determined by the student’s observable characteristics, such as home location and special needs status, and we refer to the observable characteristics as the student’s type. The second lever is how schools prioritize between different students. We assume that every student is given a number at every school, called the student’s priority at the school, and in situations of conflict the student with the higher priority is admitted. We assume that the vector of priorities for each student is drawn from a priority distribution which depends on the student’s type. The third lever is a upperbound at each school for the number of students admitted, which we call the school’s quota.

Almost all existing literature treat the menus, priority distributions, and quotas as exogenously given, over which the social planner has little control. However, in practice, these policy levers are set by the school board to induce desirable outcomes. This is because in most public school districts, the school board owns all the schools and has centralized control over menus, priority distributions, and quotas.

For example, in 2004, 2009, and 2012, Boston Public Schools tried three times to reform students’ menus in order to achieve a better balance of variety of choice, equity, and school busing costs. Previously, the city used a 3-zone plan, which divided the city into 3 geographic zones and allowed students to rank any number of schools in their zone.
This gave students about 30 schools to rank, but resulted in a staggering busing cost of about $80 million per year, since each school had to pick up children from one-third of the city (See Russell and Ebbert (2011)). Furthermore, there were perceived inequities in the distribution of schools across zones. In both 2004 and 2009, the school board proposed alternative zoning plans that divided the city into more zones and so reduced the menus and the busing burden. However, none of these plans were approved because of equity concerns. In 2013, after conducting an extensive simulation study comparing various proposed plans on a range of metrics (see Pathak and Shi (2013), Shi (2015)), the city finally approved a new proposal, called the Home-Based Plan, which gave students about 9-15 choices around their home based on proximity and standardized test scores. Furthermore, the city revised the priority distributions to no longer depend on whether students live within a one-mile radius of the school, called the school’s walk-zone. Hence, the menus and priority distributions adopted in Boston in 2013 are not exogenously given, but carefully determined policy levers to induce a desirable outcome.

This paper studies how to systematically optimize the menus, priority distributions, and quotas in the DA algorithm to maximize a given objective function, which may take into account the expected utilities of students, equity measures, socio-economic diversity, and busing costs. The input of the optimization is the school board’s objective function, a distribution for the number of students of each type, and a distribution of utilities for students of each type. (The population distribution and utility distributions can be estimated from past data.) The output is the optimized menu and priority distribution for each type of student, as well as the optimized quota for each school.

The optimization methodology is based on a new connection between stable matching and assortment planning. We first define an alternative, simpler framework of assigning students, which we call a random assortment mechanism. In this mechanism, each student is given a set of schools, which we call an assortment, and the student is assigned to his/her favorite school in the assortment. The assortment of a student may be random, but the distribution over assortments is dependent only on the type of the student. This mechanism is characterized by the probability of giving each type of student each assortment of schools, which we call assortment probabilities. We show that the DA algorithm,
given menus, priority distributions and quotas, can be equivalently expressed as a random assortment mechanism with certain assortment probabilities. This implies that when we define a new optimization problem of finding a random assortment mechanism that maximizes the school board’s objective, this problem is a relaxation of the original optimization over menus, priority distributions and quotas.

In the other direction, given the utility distribution for each type of student, we show that any random assortment mechanism can be expressed as the asymptotic behavior of the DA algorithm with certain fixed menus, priority distributions and quotas, in the limit when both the number of students of each type and the quota for each school is scaled proportionally to infinity. We call this asymptotic behavior a large-market limit, and it is based on the large-market model of stable matching of [Azevedo and Leshno (2015)]. We augment their theory by showing an equivalent characterization of large-market stable matching as a random assortment mechanism. The proof of this yields a mapping between assortment probabilities and menus, priority distributions and quotas that are equivalent in the large-market limit.

The optimization methodology is as follows: we first optimize the school board’s objective over the space of random assortment mechanisms, and then use the optimal assortment probabilities and the above mapping to construct menus, priority distributions and quotas, which we take as the output. The outputted policy levers may no longer be exactly optimal under a finite number of students, but we expect them to be approximately optimal if the number of students is large, which is the case in practice.

The optimization over random assortment mechanisms can be formulated as a convex optimization problem, which is similar in structure to the choice-based linear program (CBLP) studied in the network revenue management literature ([Liu and van Ryzin (2008)]). The convex optimization has exponentially many variables, but it can still be solved efficiently by repeatedly solving a sub-problem. This sub-problem is to find an assortment of schools for a specific type of students, such that if each student picks his/her favorite school among the assortment, and if assigning each seat of a school incurs a certain externalities, then the weighted sum of the expected utility of students of this type plus the expected externalities is maximized. We call this the socially-optimal assortment planning problem,
and show that it is a strict generalization of the revenue-maximizing assortment planning problem studied in the revenue management literature. We show that many techniques from the revenue-maximizing case can be extended to the socially-optimal case, including efficient algorithms for multinomial logit (MNL), nested logit, and Markov chain based choice models.

To demonstrate the effectiveness of this methodology, we apply it on real data from Boston Public Schools (BPS). Specifically, we take the menus and priority distributions from the Home-Based plan adopted in the 2012-2013 school assignment reform, and produce optimized menus and priority distributions that induce higher expected utilities for students while using less school busing, in terms of both the average distances traveled by students, and the average area schools have to cover to pick up students from. All of the evaluations are done by discrete simulations, and do not rely on large-market assumptions. We show that the optimized plan also performs well in aspects that do not explicitly appear in the optimization, including high chances for students to get their top choices in menu, and high chances for neighbors to go to the same school. We show that these results are robust to moderate changes in the population and utility distributions.

1.1. Related Literature

Optimization models of school assignment trace back to Clarke and Surkis (1968), Belford and Ratliff (1972), and Franklin and Koenigsberg (1973). However, these papers do not allow students to choose, but assume that the assignment is completely determined by the student’s home location. In their formulation, students are partitioned into local neighborhoods, and the goal is to find an assignment of neighborhood to schools in order to minimize the total distances traveled by students, subject to capacity, diversity, and other constraints. Sutcliffe et al. (1984) survey this literature and propose a formulation that treats the constraints as not necessarily binding but desirable goals that incur a penalty for being not fulfilled. Lemberg and Church (2000) study the dynamic problem of how to update zone boundaries smoothly across years to reflect changes in demographics and capacity. Caro et al. (2004) combine the optimization with a Geographic Information System (GIS) for easy visualization and better treatment of spatial information.
Previous literature on school choice systems have focused on finding assignment mechanisms that achieve certain axiomatic properties, and much of it supports using a variant of the DA algorithm. Abdulkadiroğlu and Sönmez (2003) show that while both the DA algorithm and another algorithm called top trading cycles (TTC) are strategyproof, they differ in their trade-off between efficiency and fairness: DA achieves Pareto-optimality among assignments that respect the priorities, while TTC achieves Pareto-optimality among all assignments. A. Abdulkadiroğlu (2005) apply this theory to the school assignment system in New York city (NYC), and their work helped NYC adopt the DA algorithm in 2004. In a follow-up paper, Abdulkadiroğlu et al. (2009) review the NYC data and argue that having a strategyproof algorithm is important to allow students to participate straightforwardly. In Boston, the algorithm before 2005 was not strategyproof. The algorithm, called the Boston Mechanism in the literature, had schools prioritize students who rank them earlier in their preferences, and gave students incentives to strategize in their preference submissions. Abdulkadiroğlu et al. (2005) argue for the adoption of a strategyproof algorithm in Boston, and their work led to Boston adopting the DA algorithm in 2005. However, Miralles (2012) and Abdulkadiroğ et al. (2011) argue that the Boston Mechanism may sometimes yield a more efficient assignment because it can better elicit the preference intensities of students. Erdil and Ergin (2008) and Abdulkadiroğlu et al. (2015) suggest ways to modify the DA algorithm to improve in efficiency.

However, the vast majority of previous literature treats the menus, priority distributions, and quotas as exogenous inputs, rather than potential policy levers. There are two strands of literature that are exceptions: the first strand considers how to implement priorities or quotas in the DA algorithm to induce diversity in the assignment, such as socio-economic or racial diversity. However, this literature focuses on how to implement a certain type of priority or quota, rather than what priorities or quotas to set. Abdulkadiroğlu and Sönmez (2003) extend both DA and TTC to allow quotas for various types of students at each school. Kojima (2012) show that there are cases in which these quotas may hurt the minority students they are intended to help. Hafalir et al. (2013) propose a way to remedy this by having soft bounds that may be violated depending on the preference of students, rather than hard bounds that always have to be satisfied. Ehlers et al. (2014) generalize.
these soft bounds to multiple types of students. Kominers and Sönmez (2015) study how to implement more complex diversity constraints in the DA algorithm by sub-dividing each school into slots and having possibly different priorities for students in each slot. Dur et al. (2013) demonstrates the impact of altering the order these slots are processed within the DA algorithm, and argue that this understanding led to the removal of the walk-zone priority in Boston in 2013.

The second strand of literature compares two specific priority distributions and discuss their impact on students. This current paper is different in that we optimize over all possible priority distributions, rather than compare only two. The two priority distributions compared in the literature both involve a combination of a strict priority between categories of students, and a tie-breaking rule within each category. The differences are in the tie-breaking rule. In single tie-breaking (STB), a single random number is given to each student, which is the same at every school. In multiple tie-breaking (MTB), a different random number is given to each student at each school, and the random numbers are independently drawn. Abdulkadiroğlu et al. (2009) compare STB and MTB by simulation using data from New York, and find that while neither of these dominate the other, STB gives more students their top choice, while MTB leaves fewer students unassigned. Ashlagi et al. (2015) and Ashlagi and Nikzad (2015) prove that similar results hold in general in a theoretical model with large number of students and seats. Arnosti (2015) show similar results in a theoretical model with a different asymptotic setup.

The most similar work to our paper is Ashlagi and Shi (2015), which is also motivated by optimizing the assignment system in Boston. However, our paper is different in three important aspects. Firstly, Ashlagi and Shi (2015) seek to optimize over all possible mechanisms, and for tractability introduce additional assumptions, which they show imply using the DA algorithm with single tie-breaking; this current paper assumes that the DA algorithm will be used, and optimize over all possible priority distributions, which include multiple tie-breaking as well as more complicated priority structures. Secondly, we develop the connection with assortment planning in greater detail, and this allows us to leverage recent algorithms from the revenue management literature to handle more complex utility distributions and constraints. Thirdly, in the empirical exercise, we consider
not only the average distance students travel, but also the area buses have to cover to pick up students, and this provides a more realistic model of transportation burden. This ability to bound the bus coverage area is only made possible through the new connection with assortment planning, and through the algorithm we develop for the socially-optimal assortment planning problem with cardinality constraints.

In terms of methodology, our work draws upon the large-market theory of stable matching developed by Azevedo and Leshno (2015) and Abdulkadiroğlu et al. (2015). The techniques used to solve the socially-optimal assortment planning are based on algorithms developed for the revenue-maximizing case, including the algorithms for the MNL model by van Ryzin and Mahajan (1999) and Rusmevichientong et al. (2010), for the nested logit model by Gallego and Topaloglu (2014) and Li et al. (2015), and for the Markov chain model by Blanchet et al. (2013) and Feldman and Topaloglu (2014b).

2. Model

There is a finite number of students to be assigned to a finite number of schools. The student population is partitioned into \( l \) types, according to observable characteristics such as residential neighborhood, special needs, test scores, etc. For each \( t \in [l] = \{1, 2, \cdots, l\} \), let \( n_t \) be the number of students of type \( t \). We assume that the vector \( \vec{n} \) is drawn from a known population distribution, denoted \( H \). Let \( m \) be the number of schools, and let the set of schools be \([m] = \{1, 2, \cdots, m\}\). We assume that there is also an outside option denoted by \( 0 \). Let \( \Omega = [m] \cup \{0\} \) be the set of all possible options. Each student is to be assigned to exactly one of these options.

For each type \( t \in [l] \), let \( I_t \) be the set of students of type \( t \). The preference of student \( i \in I_t \) for schools is described by a utility vector \( \vec{u}_i \in \mathbb{R}^{m+1} \), where \( u_{ij} \) is the student’s utility for option \( j \in \Omega \). We assume that \( \vec{u}_i, i \in I_t \), are independent and identically distributed according to a known utility distribution \( F_t \). Moreover, we assume that \( F_t \) is continuous, which implies that the probability that a student is completely indifferent between two schools is zero.

As an example of utility distributions, consider the case where

\[
u_{ij} = \bar{u}_{ij} + \epsilon_{ij} \quad \forall i \in I_t,
\]
where $\bar{u}_{tj}$ is a constant representing the average utility of type $t$ students for option $j$, and $\epsilon_{ij}$ is a random perturbation drawn i.i.d. from a Gumbel distribution with location parameter 0 and scale parameter 1. This utility distribution is called multinomial logit (MNL), and it is one of the most common ways to model discrete choice because the parameters $\bar{u}_{tj}$ are easy to estimate from data. This is the form of the utility distribution we use in our empirical application in Section 4.

2.1. The Deferred Acceptance (DA) Framework

Students are assigned to schools according to the student-proposing deferred acceptance algorithm, which we call the DA algorithm. The algorithm has three sets of policy levers controlled by the social planner, which we call menus, priority distributions, and quotas. We define the deferred acceptance framework as the set of assignment systems that use the DA algorithm with certain settings of policy levers. These policy levers are defined as follows.

- The menu $M_t \subset [m]$ for students of type $t$ is a subset of schools offered to every student of this type. The menu $M_t$ and the utility vector $\bar{u}_i$ of a student $i \in I_t$ uniquely determine his/her preference ranking of schools, which is a permutation of options in $M_t \cup \{0\}$ sorted according to decreasing utilities. We denote this preference ranking of student $i$ by $R_i$.

- The priority distribution $\Pi_t$ of type $t$ is a continuous measure on $[0, 1)^{m+1}$. Each student $i \in I_t$ is given a priority vector $\pi_i \sim \Pi_t$, where $\pi_{ij}$ is a number, which we call the priority of student $i$ at school $j$. A higher number $\pi_{ij}$ corresponds to a higher priority.

- The quota $q_j$ of school $j \in [m]$ is the maximum number of students allowed to be assigned to that school.

As described above, the menus and the priority distributions induce for each student $i$ a preference ranking $R_i$ and a priority vector $\pi_i$. Moreover, there is a quota for every school. The DA algorithm is defined in terms of these preference rankings, priority vectors, and quotas:

1. Find a student $i$ who is not yet assigned anywhere, and have him/her apply to his/her first ranked option in $R_i$.
2. If this choice is the outside option, assign the student there. Otherwise, if this choice is a school, say school $j$, then tentatively accept the student at this school.
3. If the quota of school \( j \) is not exceeded as a result of this new acceptance, go back to Step 1. Otherwise, find the tentatively accepted student \( i' \) at school \( j \) with the lowest priority \( \pi_{i'j} \), reject the student, and go to Step 2 with student \( i' \) applying to his/her next choice in \( R_{i'} \).

The algorithm terminates when every student is assigned to an option, which happens eventually as the outside option never rejects any student. The output of the algorithm is an assignment of every student to a school or to the outside option.

The algorithm was first proposed by Gale and Shapley (1962) and was first adapted for school choice by Abdulkadiroğlu and Sönmez (2003). One can show that the output does not depend on the order that students are chosen in Step 1, and that it is a dominant strategy for every student to report his/her true preference ranking (See Roth (1982) and Dubins and Freedman (1981)). This means that for each student \( i \), regardless of the preference rankings of other students \( i' \neq i \) and regardless of the priority vectors, the student’s utility is maximized when he/she reports his/her truthful ranking \( R_i \).

2.2. The Simulation Framework

The assignment outcome depends on all the parameters described so far according to the following process, which we call the simulation framework.

1. Draw \( \vec{n} \sim H \).
2. For each student \( i \in I \), draw the student’s utility vector \( \vec{u}_i \sim F_i \). Use this utility vector and the menu \( M_i \) to define the student’s preference ranking \( R_i \).
3. For each student \( i \) of type \( t \), draw the student’s priority vector \( \vec{\pi}_i \sim \Pi_i \).
4. Compute the assignment using the DA algorithm, using the students’ preference rankings \( \{R_i\} \), the priorities \( \{\pi_{ij}\} \), and the schools’ quotas \( \{q_j\} \).

We call this the simulation framework because by repeatedly simulating the above process and averaging, one can estimate to any degree of accuracy various outcome measures. The two measures we focus on are as follows: the average expected utility of students from type \( t \), which we denote by \( v_t \), and the probability that a student of type \( t \) is assigned to option \( j \in \Omega \), which we denote by \( p_{tj} \). We denote the vector of all expected utilities by \( \vec{v} \) and the matrix of all assignment probabilities by \( p \).
2.3. The Optimization Problem
Given the population distribution $H$ and the utility distributions $\{F_t\}$, the optimization problem of the school board is to set menus $\{M_t\}$, priority distributions $\{\Pi_t\}$ and quotas $\{q_j\}$ in a way to optimize a given objective function $W$, which we assume to be a function of the expected utilities $\bar{v}$ and assignment probabilities $p$. The expected utilities and assignment probabilities are related to the decision variables, $\{M_t\}$, $\{\Pi_t\}$ and $\{q_j\}$ according to the following mapping:

$$(\bar{v}, p) = DA(\{M_t\}, \{\Pi_t\}, \{q_j\}),$$

which is well defined under the simulation framework for given population distribution and utility distributions.

Under this notation, the optimization problem, which we call (OptDA), can be expressed as follows.

$$(\text{OptDA}) \quad \text{Maximize:} \quad W(\bar{v}, p)$$

subject to: $$(\bar{v}, p) = DA(\{M_t\}, \{\Pi_t\}, \{q_j\})$$

For the objective function $W$, we assume that it is non-decreasing in each component of the expected utility vector $\bar{v}$ and that it is globally concave. This level of generality allows it to incorporate a combination of diverse considerations, such as average utility of students, minimum utility of any type, capacity constraints, and convex penalty function for the average distance students travel to school (as a proxy for school busing costs). At the same time, the objective is only dependent on the expectations of utilities and assignments, which makes it tractable (as will be seen in Section 3.2). We consider a specific form of the objective function in the empirical exercise in Section 4.

2.4. Sufficiency of the Deferred Acceptance Framework
Before proceeding to the analysis, we first comment on why we confine the optimization to be within the deferred acceptance framework, instead of a more general class of assignment systems.
For example, one can imagine an alternative formulation in which the decision variables are directly the assignment outcome, which can be represented by binary variables $x_{ij}$ that indicate whether student $i$ is assigned to option $j$. The inputs to this optimization would be the students’ elicited preference over schools, and the school board’s objectives.

The issue with such an approach is that students might not have the incentive to report their true preferences, so the optimization might be based on erroneous inputs. This was a real issue in Boston before 2005, during which Boston used a non-incentive compatible assignment algorithm instead of DA. This previous algorithm, called the Boston Mechanism in the literature, prioritized satisfying students’ first choices, then only considered second choices after all first choices were filled, and third choices after second choices, and so on. This made one’s first choice very important, and parents strategized in not wasting their first choice on very popular schools, but using it on a school that they think they have a fair chance at. This resulted in prevalent gaming of preferences and a large proportion of students simply ended up unassigned. (See Abdulkadiroğlu et al. (2005).) Because of these incentive issues, the Boston school committee adopted the DA algorithm in 2005.

To resolve the incentive issue, one would need to add incentive-compatibility constraints. Assuming that the total number of students is fixed and is equal to $n$, this requires reformulating the optimization as maximizing over allocation functions $x: \mathbb{R}^{n(m+1)} \rightarrow \mathbb{R}^{n(m+1)}$, in which the input is everyone’s utility reports and the output is each student $i$’s assignment probabilities to schools. Assuming common priors $F_i$ on the utility distribution of students of type $t$, the incentive-compatibility constraints are

$$E[\tilde{u}_i \cdot \tilde{x}_i(\cdots, \tilde{u}_i, \cdots)] \geq E[\tilde{u}_i \cdot \tilde{x}_i(\cdots, \tilde{u}_i', \cdots)] \quad \forall \tilde{u}_i$$

In words, this says that the expected utility for student $i$ from reporting his true utility vector $\tilde{u}_i$, whatever that may be, must be greater than his/her expected utility from reporting any other vector $\tilde{u}_i'$. This optimization problem, being over the space of functions and having uncountably many constraints, is very complex, and the state-of-art techniques can only tractably solve this for 2 schools. (See Miralles (2012) for a paper studying a simplified version of the 2-schools case.)
For tractability, one has to assume more structure, and a natural structure is the framework already implemented in many US cities, which is choosing menus, priority distributions and quotas within the DA algorithm. This is the class of policies considered by the city committee in the 2012-2013 Boston school assignment reform (see Shi (2015) and Pathak and Shi (2013)). An advantage of using DA is that it naturally solves the incentive problem. Furthermore, Ashlagi and Shi (2015) show that under the large-market limit in which each student becomes infinitesimal, then any method of assigning students that satisfy certain natural assumptions can be represented as DA with certain menus and priority distributions. The assumptions are 1) only using preference rankings instead of preference intensities; 2) treating students symmetrically within each type; 3) incentive compatibility; and 4) no subset of students can trade assignments within their type and all improve. This shows that under certain assumptions, restricting within the deferred acceptance framework is without loss of generality.

3. Analysis

While one can evaluate by simulation the school board’s objective given any combination of menus, priority distributions and quotas, it is unclear how to optimize over these policy levers. Without simulating the entire deferred acceptance (DA) algorithm, it is difficult to express students’ utilities and assignment probabilities in terms of the policy levers, as whether a student can get into a school depends on how many others with higher priority rank it highly, which in turn depends on whether they can get into their more preferred choices.

Our approach is to approximate the original problem, (OptDA), with a simpler problem, which we call the optimal random assortment problem. In this problem, the assignment is no longer by the DA algorithm, but by a random assortment mechanism, which is to offer each type of students a randomized set of schools, and assigning each student to his/her favorite school among the offered set. Such a random assortment mechanism is characterized by the probability of offering each set of schools to each type of students, and we call these the assortment probabilities. These assortment probabilities induce a realization of the the expected utilities of each type and the assignment probabilities, which
in turn induce a realization of the school board’s objective function. We call the problem of finding assortment probabilities to maximize the school board’s objective the random assortment approximation to (OptDA).

Solving the random assortment approximation is useful for two reasons. Firstly, we show that the random assortment approximation is a relaxation to (OptDA), so its optimal objective upper-bounds the optimal objective in (OptDA). Secondly, we show that random assortment mechanisms are equivalent to the large-market approximation of stable matching due to Azevedo and Leshno (2015) and Abdulkadiroğlu et al. (2015). This implies that there is a mapping from assortment probabilities to menus, priority distributions, and quotas. Under this mapping, the corresponding random assortment mechanism can be interpreted as the asymptotic behavior of the corresponding DA algorithm, in the limit in which the number of student types and the number of schools are fixed, but the number of students of each type and the quotas of the schools are scaled up to infinity. This mapping allows us to use the optimal assortment probabilities to construct optimized menus, priority distributions, and quotas, that are approximately optimal solutions to (OptDA) when the number of students is large.

To solve the random assortment approximation, we formulate the optimization as a convex program, which can be solved by repeatedly solving a sub-problem of finding a deterministic assortment of schools for a single student type, such that if each student picks his/her favorite school among the assortment, and if assigning each seat of a school incurs a certain externalities, then the weighted sum of the expected utility of students of this type and the expected externalities is maximized. We call this the socially-optimal assortment planning problem, and show that it generalizes the revenue-maximizing assortment planning problem studied in the revenue management literature. We show that many algorithms for the revenue-maximizing case can be extended to the socially optimal case and use this to derive efficient algorithms under MNL utility distributions.

3.1. The Random Assortment Approximation

Suppose that instead of running the DA algorithm with certain menus, priority distributions and quotas, we simply offered each student of type $t$ an assortment $S \subseteq [m]$ of schools with probability $x_{tS}$, and assign each student to his/her favorite option in $S \cup \{0\}$. (Recall
that \( 0 \) represents the outside option.) We call this a random assortment mechanism. Since the assortment probabilities \( x_{tS} \) are exogenously given, any random assortment mechanism is strategyproof.

Define

\[
V_t(S) = E[ \max_{j \in S \cup \{0\}} u_{ij}],
\]

where \( u_t \sim F_t \), and

\[
P_t(j, S) = \begin{cases} 
F_t(\{u_{ij} \geq u_{ij'} \text{ for every } j' \in S \cup \{0\}\}) & \text{if } j \in S \cup \{0\}, \\
0 & \text{otherwise.}
\end{cases}
\]

\( V_t(S) \) is the expected utility of students of type \( t \) when offered assortment \( S \) and \( P_t(j, S) \) is the probability that a type \( t \) student chooses school \( j \) when offered assortment \( S \). Note that it is possible that the student chooses the outside option, so \( j \) can be 0.

The benefit of considering random assortment mechanisms is that optimizing over them can be done by solving the following convex optimization problem, which we call the random assortment convex program. (The decision variables are \( \{x_{tS}\} \). We assume also that the assortment \( S \) is restricted to be within \( \Psi_t \), which for now we define as \( \Psi_t = [m] \).

Having the more abstract form of \( \Psi_t \) simplifies the exposition in Section 3.2 and is used to approximate a non-convex objective function in our empirical application in Section 4.

\[
\text{(OptAssortments)} \quad \text{Maximize: } W(\vec{v}, \vec{p})
\]

subject to:

\[
v_t = \sum_{S \in \Psi_t} V_t(S)x_{tS}, \quad t \in [l]
\]

\[
p_{ij} = \sum_{S \in \Psi_t} P_t(j, S)x_{tS}, \quad t \in [l], j \in \Omega
\]

\[
\sum_{S \in \Psi_t} x_{tS} = 1, \quad t \in [l]
\]

\[
x_{tS} \geq 0, \quad t \in [l], S \in \Psi_t
\]

While there are exponentially many decision variables, we can use standard techniques in convex optimization to reduce this to repeatedly solving simpler, tractable problems, which we do in Section 3.2. Before solving this optimization, we will elucidate the connection between this problem, (OptAssortments), with the original optimization, (OptDA).
Section 3.1.1 shows that (OptAssortments) is a relaxation of (OptDA). Section 3.1.2 interprets (OptAssortments) as a large-market approximation of (OptDA), and Section 3.1.3 use this relationship to map the solution of (OptAssortments) back into a feasible solution of (OptDA), which is the final output of our optimization.

3.1.1. Random Assortment as Relaxation

The first interpretation of (OptAssortments) is that it is a relaxation of (OptDA). This comes from the following theorem, which says that in the DA algorithm, each student is being assigned his/her favorite school from a set which we call his/her accessible assortment. The accessible assortment is independent of the student’s own preferences, but possibly dependent on the number of students of each type, the preferences of all other students and the priorities of every student.

**Definition 1.** For a student $i$, define his/her accessible assortment as the set of options he/she can be assigned to in the student-proposing deferred acceptance (DA) algorithm if he/she chooses that option first, and ranks the outside option as more preferred than any other option.

**Theorem 1.** In the student-proposing deferred acceptance algorithm, every student is assigned to his/her favorite school among his/her accessible assortment.

**Proof of Theorem 1.** The desired result is implied by the strategyproofness of student-proposing DA for the students (Roth (1982) and Dubins and Freedman (1981)). To see this, suppose that in the student-proposing DA, student $i$ is rejected by schools $j_1, \cdots, j_{k-1}$ and finally assigned to option $j_k$.

By the strategyproofness of student-proposing DA, if $i$ had ranked $j_k$ first, $i$ would still be matched to $j_k$. This is because if $i$ cannot get $j_k$ by ranking it first, but can get it by ranking other schools first, then the student has incentives to misreport preferences if $j_k$ happened to be his/her true first choice.

It suffices to show that none of the previous options, $j_1, \cdots, j_{k-1}$ are in the accessible assortment. This again follows from strategyproofness, because if $i$ can get any of these schools by ranking it first, then $i$ would have an incentive to deviate because that improves upon $i$’s current assignment of $j_k$. Q.E.D.
In particular, the independence between a student’s accessible assortment and own preferences imply that if $x_{ts}$ is the probability that the accessible assortment of students of type $t$ is $S$, then the expected utility of students of type $t$ in the DA algorithm is exactly as in Equations \[4] of (OptAssortments), and the assignment probabilities are as in Equation \[5]. Furthermore, the probabilities $x_{ts}$, must sum to one for each type, so satisfy Equation \[6]. This implies that (OptAssortments) with $\Psi_t = [m]$ is a relaxation of (OptDA), so its optimal objective must upper-bound the optimal objective of (OptDA).

### 3.1.2. Random Assortment as Large-Market Approximation

An alternative interpretation of (OptAssortments) is that it is solving for the optimal menus, priority distributions and quotas in the large-market approximation of DA from Azevedo and Leshno (2015). This relationship allows us to map the optimal assortment probabilities back into menus, priority distributions and quotas, which can be interpreted as approximately optimal solutions to the original problem, (OptDA). In the following definition, we adapt the machinery developed by Azevedo and Leshno (2015) into our setting.

**Definition 2.** Given the number of students of each type $\{n_t\}$ and the utility distributions $\{F_t\}$, a large-market stable matching is a tuple $(\{M_t\}, \{\Pi_t\}, \{q_j\}, \{h_j\})$, such that $M_t$ and $\Pi_t$ are the menu and priority distribution of type $t$, and $q_j$ is the quota of school $j \in [m]$. The additional parameters are priority cutoffs $h_j \geq 0$ for every option $j \in \Omega$, which is defined to be zero for the outside option $j = 0$. This stable matching represents a scenario in which every student $i$ of type $t$ receives a priority vector $\vec{\pi}_i \sim \Pi_t$, and is assigned to his favorite option $j$ within $M_t \cup \{0\}$ for which his priority meets the cutoff: $\pi_{ij} \geq h_j$. The priority cutoffs are related to the quotas in that for schools whose quota is not filled, the priority cutoff is restricted to be zero.

The intuition behind this large-market approximation is as follows: consider multiplicatively scaling up both the number of students of each type $\{n_t\}$ and the quotas $\{q_j\}$ by the same multiplicative parameter $\gamma$, then the stochasticity in the preference and priority draws is averaged away, and the priority needed to get into each school $j$ converges to a constant, represented by the cutoff $h_j$. This is zero for schools with unfilled quota, as
the DA algorithm would accept anyone who applies to such schools. For schools in which
the quota is filled, this is the lowest priority among the accepted students, which is the
priority one needs to beat to be assigned in the DA algorithm. A full derivation of this
large market limit and its properties can be found in Azevedo and Leshno (2015).

Under this large-market stable matching model, one can analogously define the accessible
assortment as the set of options that are in menu and for which one’s priority meets the
cutoff. For a student of type \( t \), the probability that the accessible assortment is \( S \) is exactly

\[
x_{tS} = \begin{cases} 
\Pi_t(\{\pi_{ij} \geq h_j \forall j \in S \text{ and } \pi_{ij} < h_j \forall j \in M_t \setminus S\}) & \text{if } S \subseteq M_t, \\
0 & \text{otherwise.} 
\end{cases} 
\]  

(8)

Using the proof of part (1) of Theorem 1 of Azevedo and Leshno (2015) (page 5 in their
appendix) and the proof of part (1) of their proposition 3 (page 16 in their appendix), one
can show the following.

**Proposition 1.** Given the number of students of each type \( \{n_t\} \) and utility distribu-
tions \( \{F_t\} \). Assume that each \( F_t \) satisfies full-support, which means that for every per-
mutation of the options \( \Omega = [m] \cup \{0\} \), utilities drawn according to \( F_t \) would induce this
preference ranking with positive probability. Then for every combination of menus \( \{M_t\} \),
priority distributions \( \{\Pi_t\} \), quotas \( q_j \), there exists a unique set of cutoffs \( \{h_j\} \) for which
\((\{M_t\}, \{\Pi_t\}, \{q_j\}, \{h_j\})\) is a large-market stable matching.

Furthermore, consider any sequence of finite markets indexed by \( k \), with \( n^k_t \) students of
type \( t \), and a quota of \( q^k_j \). Then as \( k \to \infty \), if \( \frac{n^k_t}{k} \to n_t \) and \( \frac{q^k_j}{k} \to q_j \), then the distribution
of the accessible assortment under the DA algorithm (see Definition 7) converges to the
distribution of accessible assortment under the above large-market model, with assortment
probabilities given by Equation 8.

This formally establishes that the large-market approximation is well defined and
approximates the outcome of the DA algorithm when the number of students of each type
and the quota of each school is scaled up to infinity. Having connected the original DA
framework to large-market stable matchings, the following theorem connects large-market
stable matchings with random assortment mechanisms.
THEOREM 2. Fix the number of students of each type \( \{n_t\} \) and utility distributions \( \{F_t\} \). Any large-market stable matching \((\{M_t\}, \{\Pi_t\}, \{q_j\}, \{h_j\})\) can be equivalently represented as a random assortment mechanism with certain assortment probabilities \( \{x_{tS}\} \). Conversely, any random assortment mechanism with assortment probabilities \( \{x_{tS}\} \) can be represented as a large-market stable matching with certain parameters \((\{M_t\}, \{\Pi_t\}, \{q_j\}, \{h_j\})\).

Proof of Theorem \[2\] The first direction is immediate. Given \( \{M_t\}, \{\Pi_t\}, \{q_j\}, \{h_j\} \), define \( x_{tS} \) as in equation \[8\] The random assortment mechanism with these assortment probabilities assigns every student to their favorite school in menu for which their priority meets the cutoff.

The second direction takes a little more work. Given the assortment probabilities \( x_{tS} \), let

\[
M_t = \bigcup_{S: x_{tS} > 0} S \quad \text{for all } t \in [l],
\]

\[
h_j = \frac{1}{2} \quad \text{for every school } j \in [m],
\]

\[
q_j = \sum_{t=1}^{l} n_t \sum_{S \ni j} P_t(j, S) x_{tS} \quad \text{for every school } j \in [m].
\]

For the priority distributions \( \{\Pi_t\} \), define for each student two random variables, \( S_i \) and \( \delta_i \), where \( S_i \subseteq [m] \) is distributed according to \( x_{tS} \) and \( \epsilon_i \sim \text{Uniform}[0,1] \). Let each student’s priority vector \( \vec{\pi}_t \) be such that

\[
\pi_{ij} = \frac{1}{2} (\mathbb{1}(j \in S_i) + \delta_i).
\]

This priority distribution is constructed so that for each student \( i \) of type \( t \), the set of schools for which the student’s cutoff is above the the cutoff of \( h_{j/2} \) is exactly \( S_i \), which is distributed according to the assortment probability \( x_{tS} \). The definition of quotas make it so that every school is exactly filled, which removes any restrictions on the cutoffs. Q.E.D.
3.1.3. The Optimized Menus, Priority Distributions, and Quotas

The proof of Theorem 2 yields a possible mapping of assortment probabilities \( \{x_{iS}\} \) into menus, priority distributions, and quotas. While this mapping can be applied to general assortment probabilities, the following simpler mapping works in the special case when the assortments for each type \( C_t = \{S : x_{iS} > 0\} \) are nested. The advantage of the following mapping is that it is simpler to implement, being the sum of a constant term \( h_{tj} \) and a random number \( \epsilon_i \). Moreover, the final assignment is also guaranteed to be Pareto optimal within each type, meaning that students cannot trade assignments within their type and all improve. This form of priorities is used also in Ashlagi and Shi (2015). The mapping has the same menus, cutoffs, and quotas as in Equations 9, 10, and 11, but the priorities become:

\[
\pi_{ij} = \frac{1}{2} (h_{tj} + \epsilon_i) \tag{14}
\]

where \( \epsilon_i \sim \text{Uniform}[0,1] \) as before and

\[
h_{tj} = \sum_{S \ni j} x_{iS}.
\]

It is this mapping that we will apply in Section 4.

3.2. Solution Method to the Random Assortment Convex Program

The random assortment convex program as formulated in Section 3.1 (OptAssortments), has exponentially many decision variables \( x_{iS} \). A standard approach for such problems is simplicial decomposition, which is a generalization of the column generation technique from linear programming. This is an iterative algorithm that maintains for each type a subset \( C_t \subseteq \Psi_t \). From any feasible solution of (OptAssortments), we initialize each \( C_t \) to be the set of assortments \( S \) for which \( x_{iS} > 0 \) in the feasible solution.

The algorithm is iterative, and the sets \( C_t \) expand after each iteration. In each iteration, we first solve a master problem, which is the same as the original formulation of (OptAssortments) except that we replaces \( \Psi_t \) by the smaller set \( C_t \), which greatly simplifies the optimization if the cardinalities of the \( C_t \)'s are small.
Given the optimal solution of the master problem, we compute a super-gradient \((\vec{\alpha}, r)\) of the concave objective \(W(\vec{v}, p)\), such that \(\alpha_t\) is the component of the super-gradient for \(v_t\) and \(r_{tj}\) is the component for \(p_{tj}\). Note that \(\alpha_t \geq 0\) by our assumption that the objective is non-decreasing in every component of \(\vec{v}\).

The sub-problem for each type \(t\) is to find an assortment \(S \in \Psi_t\) that optimizes the following linear objective:

\[
\max_{S \in \Psi_t} \alpha_t V_t(S) + \sum_{j \in \Omega} r_{tj} P_t(j, S),
\]

where \(V_t(S)\) is the value of assortment \(S\) for type \(t\) students as in Equation 2, and \(P_t(j, S)\) is the probability of preferring option \(j\) the most among \(S \cup \{0\}\) as in Equation 3.

For each type \(t\), let the optimal solution to the above sub-problem be \(S^*_t\). We append \(S^*_t\) to \(C_t\) for every type \(t\), and iterate again to resolve the master problem. The algorithm terminates if the optimal objective of the master problem does not improve between two successive iterations. See Von Hohenbalken (1977) for the development of the theory of simplicial decomposition and proof of correctness. For a more recent exposition, see Chapter 4 of Bertsekas (2015).

### 3.2.1. The Key Sub-Problem: Socially-Optimal Assortment Planning

The sub-problem (Equation 15) has the following interpretation: find for each type \(t\) a deterministic assortment \(S\) within a constraint set \(\Psi_t\) to maximize the weighted sum of two terms: the expected utility of type \(t\) and the externalities on others. The expected utility is \(V_t(S)\) and is multiplied by parameter \(\alpha_t \geq 0\), which represents how much to weigh the utility of this type. There is an externality of \(r_{tj}\) of assigning each student of type \(t\) to each option \(j \in \Omega\), so the average externalities from assortment \(S\) is \(\sum_{j \in \Omega} r_{tj} P_t(j, S)\). For the MNL model,

\[
V_t(S) = \log \left( \sum_{j \in S \cup \{0\}} \exp(\bar{u}_{tj}) \right) + \gamma_{\text{Euler}},
\]

where \(\gamma_{\text{Euler}} = .577\ldots\) is the Euler constant, and

\[
P_t(j, S) = \begin{cases} \frac{\exp(\bar{u}_{tj})}{\sum_{j' \in S \cup \{0\}} \exp(\bar{u}_{tj'})} & \text{if } j \in S \cup \{0\}, \\ 0 & \text{otherwise.} \end{cases}
\]
If \( \alpha_t = 0 \), then this is exactly the revenue-maximizing assortment planning problem studied in the revenue management literature. In that problem, \( r_{tj} \) is interpreted as the unit revenue of assigning each person of type \( t \) to option \( j \). With positive \( \alpha_t \), we call this the *socially-optimal assortment planning* problem, since the objective can be interpreted as the weighted sum of consumer welfare and revenue.

In the following, we extend classical results for the MNL model (defined in Section 2) from the revenue-maximizing case to the socially-optimal case. This gives us efficient algorithms to solve the sub-problem (Equation 15) under MNL utilities, which allows us to efficiently solve the assortment planning convex program, (OptAssortments). For ease of exposition, for the remainder of this section, we drop the suffix \( t \) in \( \alpha_t, \Psi_t, V_t, P_t, r_{tj}, \) and \( \bar{u}_{tj} \), since everything in this section pertains to a fixed type \( t \). For each option \( j \in \Omega \), define \( w_j = \exp(\bar{u}_j) \). Plugging in the formula for \( V(S) \) and \( P(j,S) \) into Equation 15, the problem we need to solve is

\[
\max_{S \in \Psi} \alpha \log \left( \sum_{j \in S \cup \{0\}} w_j \right) + \frac{\sum_{j \in S \cup \{0\}} r_j w_j}{\sum_{j \in S \cup \{0\}} w_j}.
\]  

(16)

In the revenue management literature in which \( \alpha = 0 \), a classical result due to [van Ryzin and Mahajan (1999)] is that when \( \Psi = [m] \), the optimal assortment is revenue-ordered, which means in our setting that whenever the assortment contains any option \( j \), it must contain any option \( j' \) with better unit externalities, \( r_{j'} > r_j \). [Rusmevichientong et al. (2010)] show that with cardinality constraints, in which \( \Psi = \{ S \subseteq [m] : |S| \leq k \} \) for some parameter \( k > 0 \), the revenue-ordered property no longer holds, but one can still compute the optimal assortment efficiently as follows. Define function

\[
f(\lambda) = \max_{S \in \Psi} \left\{ \sum_{j \in S \cup \{0\}} w_j (r_j - \lambda) \right\}.
\]  

(17)

This is the maximum of finitely many decreasing linear functions, so is convex and decreasing. There exists a unique \( \lambda^* \) such that \( f(\lambda^*) = 0 \). [Rusmevichientong et al. (2010)] show that the optimal objective to the assortment planning problem (Equation 16 with \( \alpha = 0 \)) is exactly \( \lambda^* \) and an assortment \( S^* \subseteq \Psi \) is an optimal solution if and only if

\[
S^* \in \arg\max_{S \in \Psi} \sum_{j \in S \cup \{0\}} w_j (r_j - \lambda^*).
\]
This generalizes the first result because with \( k = \infty \), \( \Psi = [m] \), the above implies that an assortment is revenue-optimal if and only if it contains every option with \( r_j > \lambda^{*} \) and no option with \( r_j < \lambda^{*} \).

This result yield a polynomial-time algorithm to compute the optimal assortment because \( f(\lambda) \) can be computed efficiently under cardinality constraints. This is because at any \( \lambda \), \( f(\lambda) \) is simply the sum of the largest \( k \) terms \( z(\lambda) = w_j(r_j - \lambda) \), as long as all of them are positive. (If some of the largest \( k \) terms are negative at \( \lambda \), then we do not include them in the sum.) Treating \( z(\lambda) \) as a linear function with \( \lambda \) as the x-axis, we see that the identity of the included terms is fixed within any region in which the lines do not cross one another or cross the x-axis. Since there are at most \( O(m^2) \) such crossing points, one can compute a piece-wise linear representation of \( f(\lambda) \) by first sorting the intersection points, and computing the linear representation within each region between successive intersection points. This piece-wise linear representation has at most \( O(m^2) \) pieces and can be computed in \( O(m^2 \log(m)) \) time. (See Rusmevichientong et al. (2010).)

We generalize these results to the case with positive \( \alpha \) using the following theorem.

The intuition is to use quasi-convexity and convex duality to reformulate the original optimization as a simpler, one-dimensional optimization.

**Theorem 3.** With \( \alpha > 0 \), define

\[
\Lambda^{*} = \arg \max_{\lambda \in \mathbb{R}} \{ f(\lambda) \exp\left( \frac{\lambda}{\alpha} \right) \}, \tag{18}
\]

then an assortment \( S^{*} \) is an optimal solution to the socially-optimal assortment planning problem with MNL utilities (Equation 16) if and only if

\[
S^{*} \in \bigcup_{\lambda^{*} \in \Lambda^{*}} \arg \max_{S \in \Psi} \sum_{j \in S \cup \{0\}} w_j(r_j - \lambda^{*}). \tag{19}
\]

**Proof of Theorem 3.** Define \( x(S) = \sum_{j \in S \cup \{0\}} w_j \), \( y(S) = \sum_{j \in S \cup \{0\}} r_j w_j \), \( D = \{(x(S), y(S)) : S \in \Psi\} \), and \( g(x, y) = \alpha \log(x) + \frac{y}{x} \). Since there is a correspondence between points in \( D \) and assortments, the original problem (Equation 16) can be formulated as

\[
\max_{(x, y) \in D} g(x, y) \tag{20}
\]
Note that in the domain $R = (0, \infty) \times (-\infty, \infty)$, the function $g(x, y)$ is quasi-convex, continuous, and strictly increasing in $y$. This allows us to use the following lemma, which exploits these properties of $g$ to reformulate the problem as a simpler, one-dimensional optimization.

**Lemma 1.** Let $R$ be an open convex subset of $\mathbb{R}^2$ and let $g(x, y) : R \rightarrow \mathbb{R}$ be a quasi-convex, continuous function that is strictly increasing in $y$. Let $D$ be a finite set of points from $R$. Define

$$f(\lambda) = \max_{(x, y) \in D} \{ y - \lambda x \},$$

$$A(\lambda) = \arg \max_{(x, y) \in D} \{ y - \lambda x \},$$

$$h(\lambda, f) = \inf_{(x, y) \in R} \{ g(x, y) : y = f + \lambda x \}.$$ (23)

Then

$$\max_{(x, y) \in D} g(x, y) = \sup_{\lambda \in \mathbb{R}} h(\lambda, f(\lambda)).$$ (24)

Moreover, the supremum on the right is attainable, and $(x^*, y^*)$ is an optimal solution of the left hand side if and only if $(x^*, y^*) \in A(\lambda^*)$ for some optimal solution $\lambda^*$ of the right hand side.

The proof of this Lemma is in Appendix F.

Plugging in the specific form of $g$, we find that the $h$ function in our case is $h(\lambda, f) = \alpha \log(\frac{\lambda}{\alpha}) + \lambda + \alpha$. Optimizing this with respect to $\lambda$ is equivalent to optimizing $f \exp(\frac{\lambda}{\alpha})$. This implies the desired result. Q.E.D.

Theorem 3 is generalized in Appendix G for any Generalized Extreme Value (GEV) utility distribution, which includes nested logit, and multi-level nested logit utility distributions.

This theorem gives us a polynomial-time algorithm to solve problem 16 if we can find in polynomial-time a piece-wise linear representation of $f(\lambda)$ with polynomially-many pieces. This is because for each linear segment, $a - b\lambda$, the optimal solution to Equation 18 can be found analytically, by checking the end points of the line segment and by checking $\lambda = \frac{a}{b(1+\alpha)}$. 
This yields a polynomial-time algorithm for problem [16] if $\Psi$ is a matroid, which is defined as follows and generalizes cardinality constraints.

**Definition 3.** A matroid over ground set $[m]$ is a set $\Psi$ of subsets of $[m]$ such that 1) $\emptyset \in \Psi$; 2) if $S' \subset S \subseteq [m]$ and $S \in \Psi$, then $S' \in \Psi$; and 3) if $S, S' \in \Psi$ and $|S'| > |S|$, then there exists $j \in S' \setminus S$ such that $S \cup \{j\} \in \Psi$.

A well-known result in combinatorial optimization is that when optimizing a sum over a matroid as in Equation [17], the greedy algorithm is optimal. This implies that in computing $f(\lambda)$ of Equation [17] for each $\lambda$, we can go through options $j$ in decreasing order of $w_j(r_j - \lambda)$ and greedily include $j$ as long as this does not violate the matroid constraint. (Precisely speaking, we compute a set $S$ by initializing it as the empty set, and iteratively going through options $j$ sorted in this way, and adding elements to $S$ as long as this does not violate $S \in \Psi$. After this terminates, we define $f(\lambda) = \sum_{j \in S \cup \{0\}} w_j(r_j - \lambda)$.) Since the construction of $S$ only depends on the order of the lines $z(\lambda) = w_j(r_j - \lambda)$, there are at most $O(m^2)$ segments and $f(\lambda)$ can be calculated in $O(m^2 \log(m))$ time (assuming that checking whether a set $S$ satisfies $S \in \Psi$ takes constant time).

4. Empirical Application

We apply the optimization framework on data from Boston Public Schools (BPS) to yield improved menus and priority distributions for elementary school assignment. The motivation is that Boston launched a reform to alter their menus and priority distributions in 2012-2013. (The quotas were based on capacity limits and were considered fixed and not a part of the reform.) The menus and priority distributions before the reform constituted what is called the 3-Zone plan, which divided the city into three geographic zones. The menu of a student was the union of any school within the zone containing the student’s home, and any school within a one-mile radius of the student’s home. This one-mile radius is called the student’s walk-zone, and if the student travels outside the walk-zone, then the school board is obligated to send a school bus to pick up the student.

Because of these large menus, each school had to pick up children from about one-third of the city, and this resulted in Boston paying over 80-million dollars a year in 2012, which

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1 As a comparison, the per capita spending on busing in 2012 for Boston is over 3 times the national average and 30% higher than in New York City. (Source: US Census Public-Secondary Education Finance Data 2012.)
is about 10% of its budget, on busing. (See Russell and Ebbert (2011).) The menus and priority distributions after the reform was called the Home-Based plan, which was designed to reduce the busing burden of the city by reducing the menus, while giving students the schools they want the most by constructing a menu centered around the student’s home, based on test-scores, capacities, and special programs (See Appendix A for a precise definition of the Home-Based plan). The plan also compensated families living in worse neighborhoods with more choices.

The city decided to adopt the Home-Based plan after reviewing detailed simulation results. The original analysis is found in Pathak and Shi (2013) and Shi (2015), and uses a similar setup as Section 2. In this section, we use our optimization framework to evaluate how much room for improvement there is for the Home-Based plan. Specifically, we use the same busing allowance as in Home-Based, but optimize the sum of the average utilities of students and the minimum utility of any neighborhood. The output of the optimization are menus and priority distributions, which can be incorporated within the DA algorithm similar to the 3-Zone and Home-Based plan. We find that substantial improvement is possible, as the optimized plan not only achieves better expected utilities, but also improves students’ chances of getting top choices in menu and improves students’ chances of going to school with their neighbors.

The distributional assumptions are in Section 4.1. The outcomes of interest are described in Section 4.2. We apply the optimization framework in Section 4.3 and discuss the results in Section 4.4.

4.1. Distributional Assumptions

In this section, we outline the distributional assumptions behind both our simulation results and our optimization. These assumptions are the same as in the empirical exercise in Ashlagi and Shi (2015). As in Section 2 we specify a partitioning of students into types, a population distribution, and a utility distribution for each type. Since the quotas was not part of the reform, we will also specify the quotas for each school. These parameters are based on BPS data from 2010-2013 for grade Kindergarten-2, which is the main entry grade to the elementary school system.
4.1.1. Student Types

We partition students into types based on geographic location alone. The BPS dataset partitions Boston into 868 geographic tracts, which we will call neighborhoods. We define each neighborhood to be a type.

The data also groups the 868 neighborhoods into 14 larger regions, which are based on natural divisions of the city. (For example, downtown is a region by itself.)

4.1.2. Population Distribution

Based on projections from previous years, we model $n_t$, the number of students from neighborhood $t$, as follows. Define a normal random variable with mean 4294 and standard deviation 115. This represents the total number of applicants and is estimated from historic data from 2010-2013. To accommodate medium-scale regional variations, we generate an independent normal random variable for each of the 14 regions, which represents the proportion of students who come from this region. The means and standard deviations as in Table 3 of Appendix B. The total number of students of each region is the product of the overall normal variable with the region-specific term, rounded to the nearest integer. Having computed this regional total, we sample the neighborhood $t$ of each student based on the historic density in 2010-2013. This process induces random variables $n_t$ for each neighborhood $t$, which are positively correlated both across the city and within each region.

4.1.3. Utility Distributions

As in Ashlagi and Shi (2015), we estimate a MNL model of students’ preferences. Let $i$ be a student of neighborhood $t$ and let $j$ be a school. Assume that utilities take the form

$$
    u_{ij} = \bar{u}_{tj} + \beta \epsilon_{ij}, \\
    \bar{u}_{tj} = Q_j - \text{Distance}_{tj} + \gamma \cdot \text{Walk}_{tj}.
$$

The data in the above equations are $\text{Distance}_{tj}$, and $\text{Walk}_{tj}$, and the parameters are $Q_j$, $\gamma$ and $\beta$. $\text{Distance}_{tj}$ is the walking distance from the centroid of neighborhood $t$ to school $j$ according to Google Maps. $\text{Walk}_{tj}$ is an indicator that school $j$ is within the walk-zone.

$^2$ It is also possible to consider other differences across students, such as race, older siblings, special education needs, and language learning needs. However, for clarity of analysis, we focus on the geographic aspects in this exercise.
Table 1  Parameters of the MNL model, estimated from preference data from 2013. The values can be interpreted in units of miles (how many miles a student is willing to travel for one unit of this variable).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_j$</td>
<td>0–6.29</td>
<td>Quality of schools. For a school of $\Delta Q$ additional quality, holding fixed other components, a student would be willing to travel $\Delta Q$ miles further. The value for each school is graphically displayed in Figure 1b of Appendix B.</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.86</td>
<td>Additional utility for going to a school within the walk-zone.</td>
</tr>
<tr>
<td>$\beta$</td>
<td>1.88</td>
<td>Scale parameter of the Gumbel term.</td>
</tr>
</tbody>
</table>

of neighborhood $t$. $Q_j$ is a school-specific fixed effect capturing overall school popularity, and we call this the inferred quality of school $j$. $\beta$ is the size of the perturbation term in the utility (recall that $\epsilon_{ij}$ is Gumbel distributed with location parameter 0 and scale parameter 1). $\gamma$ is a coefficient for living within one-mile.

Note that the normalization in the above is different from that in Section 2. Instead of normalizing the scale of the Gumbel term to one, we allow a scale of $\beta$, while normalizing the distance coefficient to one. Either normalizations induce the same choice behavior, and the difference is entirely in interpretation. We estimate the parameters $Q_j$, $\gamma$ and $\beta$ from rank-order preference rankings from the past, using the maximum likelihood technique of Hausman and Ruud (1987). The estimates are shown in Table 1. For the inferred qualities of schools, we plot them on a map in Figure 1b in Appendix B. We normalize the lowest $Q_j$ to zero.

Note that this model seeks to estimate from students’ preference rankings also their preference intensities. The logic behind this is as follows: assuming that the differences in how students from different neighborhoods rank schools can entirely be explained by geography, then we can infer students’ preference strengths by observing how quickly they trade these preferences for distance. For example, suppose that students generally prefer school A over school B. In neighborhoods equidistant from the two schools, then we would expect more students rank A before than B. However, as we move through neighborhoods going closer to B, we may see students preferring B more. By observing the speed at which their preference distribution change, we can have a rough estimate of the strength of their preferences.

---

3 In the estimation, we use the normalization in which the scale of the idiosyncratic term is one. After estimating the parameters, we re-normalize to make the distance coefficient one.
It is possible also to add non-linear terms of distance as well as interactions between students’ race and income and the school’s demographics and test-scores, as in Shi (2015) and P. and Shi (2015). P. and Shi (2015) also compare the above model to a mixed MNL model, which allows random coefficients, and they show that the models perform similarly in prediction accuracy in the Boston data.

4.1.4. Schools and Quotas
There are \( m = 77 \) schools in our dataset, each of which has a capacity constraint, which we model as a hard quota \( \bar{q}_j \). In contrast to our model in Section 2, the quotas are given inputs rather than decision variables in the optimization. Figure 1a plot the school quotas and locations.

One complication with hard quotas is that the school board is obligated by law to assign every applicant, and in practice, the school board may open up new classrooms or slightly increase class sizes in order to accommodate unassigned students. In Boston, this is often done manually after the main assignment has taken place. In the Home-Based plan, there is a set of 19 schools which are called “capacity schools,” which in the original proposal were intended to be schools that BPS can expand capacities at to accommodate excess demand. So many of the unassigned students may be later assigned to one of these capacity schools.

For concreteness, we assign each neighborhood a default school, which is the closest capacity school. Moreover, we treat the default school for a type not as a regular school option, but as the outside option of that type, so assignment to one’s outside option is synonymous with assignment to one’s default school. This implies that students assigned to the default school are not counted against the quota of the default school, which guarantees that every student can at least be assigned to his/her default school. For each school \( j \), define \( T_j \) as the types for which the school \( j \) is not the type’s default school. These are the types on which the quota \( \bar{q}_j \) applies.

4.2. Outcomes of Interest
Given any setting of menus and priority distributions, we use the quotas from Section 4.1.4 and compute the following set of metrics. The first and second metrics measure the transportation burden for the city. The third measures efficiency and equity in terms of students’
welfare. The fourth measures predictability, and the fifth measure community cohesion. The first three outcomes are the most important, and they will directly enter the optimization in Section 4.3. The last two are secondary, and do not enter the optimization.

1. **Average busing distance**: Define $b_{tj} = \text{Distance}_{tj}(1 - \text{Walk}_{tj})$. This is the distance between school $j$ and neighborhood $t$ if bus transportation is needed, and zero otherwise. The average busing distance is the average of $b_{tj}$ among all students.

2. **Average bus coverage area**: This is the area the average school has to cover to pick up children from. Let $\text{Area}_t$ be the size of neighborhood $t$, and let $\text{Walk}_t$ be the set of schools within the one-mile walk zone of neighborhood $t$. The average bus coverage area is

$$\frac{1}{m} \sum_{t \in [l]} |M_t \setminus \text{Walk}_t| \text{Area}_t.$$  \hfill (25)

3. **Expected utilities of neighborhoods**: For each neighborhood $t$, this is the expected utility of a student from neighborhood, without conditioning on the student’s own utility draw. This is exactly the $v_t$ as defined in Section 2.2. To aggregate the results from all 868 neighborhoods, we compute:

a) Weighted average: $\frac{\sum_{t \in [l]} E[n_t] v_t}{\sum_{t \in [l]} E[n_t]}$.

b) 10th percentile from the bottom.

c) Minimum $v_t$.

4. **Probability of getting top choice in menu**: This is estimated from the empirical frequency of getting one’s top choice within menu from the simulation framework in Section 2.2. Similarly, we compute the chance of getting one of the top three choices.

5. **Number of neighbors co-assigned**: For each neighborhood $t$, we compute the empirical average across many simulations of the number of peers students from this neighborhood have at their assigned school who live less than half mile away according to Google Maps walking distance. We report the median across neighborhoods. A similar metric for community cohesion is studied in Ashlagi and Shi (2014).

Note that we use distance and bus coverage as proxies for busing costs. This is because these were the metrics used to measure busing burden by the city committee during the 2012-2013 reform (See Pathak and Shi (2013)). A more direct estimate is also very difficult to compute as bus routes depend on previous year’s assignments and institutional
constraints. Furthermore, the actual bus routes used by BPS is computed by an independent company using proprietary software, which we have no access to.

4.3. Details in Applying the Random Assortment Approximation to Boston

We formulate the optimization as finding menus and priority distributions that, when evaluated using the simulation framework of Section 2.2 with the quotas and distributional assumptions in Section 4.1, maximize the sum of the weighted average and minimum expected utilities of neighborhoods, subject to staying within the Home-Based plan in the average busing distance and bus coverage area.

One difficulty in applying the optimization framework in Section 3 is that the bus coverage area in Equation 25 cannot be directly incorporated into a concave objective $W$ in terms of utilities and assignment probabilities. As a heuristic to overcome this, we take advantage of the flexibility of having arbitrary set constraints $\Psi_t$ in the formulation of (OptAssortments) in Section 3.1. For each neighborhood $t$, we limit the assortment to have at most $k$ schools outside of the one-mile walk-zone, with $k$ being a parameter to be determined later. Precisely speaking, let

$$\Psi_t = \{ S \subseteq [m] : |S \setminus \text{Walk}_t| \leq k + \text{Walk}_{t0} - 1 \},$$

where $\text{Walk}_t$ is the set of schools within one-mile, and $\text{Walk}_{t0}$ is an indicator for whether the default school of neighborhood $t$ is within the walk-zone. (Recall from Section 4.1.4 that we model the default school as the outside option, so in order to have $k$ be the limit including the default school, we add the term $\text{Walk}_{t0} - 1$.) This cardinality constraint in the assortment is intended to limit the cardinality of the menus, and so induce a low bus coverage area.

After applying this heuristic, the optimization can now be expressed in a form as in (OptAssortments). Let $\bar{q}_j$ be the fixed quota from section 4.1.4, which only applies to neighborhoods in the subset $T_j$. Let $B$ be a limit on the average busing distance. For the MNL model in Section 4.1.3, the assortment valuations and assignment probabilities are

$$V_t(S) = \beta \log \left( \sum_{j \in S \cup \{0\}} \exp(\bar{u}_{ij}/\beta) \right)$$

(27)
Using these parameters, the random assortment convex program for Boston becomes as follows:

(OptAssortBoston)

\[
\text{Maximize: } \quad \text{AvUtility} + \text{MinUtility} \\
\text{subject to: } \quad \sum_{t \in [l]} E[n_t] \sum_{t \in [l]} \sum_{S \in \Psi_t} E[n_t] v_t = \text{AvUtility} \\
\quad v_t \geq \text{MinUtility}, \quad t \in [l] \\
\quad \sum_{i \in T_j} \sum_{S \in \Psi_t} E[n_t] p_{ij} \leq \bar{q}_j, \quad j \in [m] \\
\quad \sum_{S \in \Psi_t} V_t(S)x_{tS} = v_t, \quad t \in [l] \\
\quad \sum_{S \in \Psi_t} P_t(j, S)x_{tS} = p_{ij}, \quad t \in [l], j \in [m] \cup \{0\} \\
\quad \sum_{S \in \Psi_t} x_{tS} = 1, \quad t \in [l] \\
\quad x_{tS} \geq 0, \quad t \in [l], S \in \Psi_t
\]

This is efficiently solvable using the technique in Section 3.2, because \(\Psi_t\) in Equation 26 describes a matroid (see Definition 3). The solution depends on two parameters, an allowance \(B\) for average busing distance, and an allowance \(k\) for the cardinality of assortments outside of one’s walk-zone, and we tweak these parameters by trial and error so the corresponding menus and priority distributions from Section 4.3.2 use less busing distance and coverage area than the Home-Based Plan. The final choice of these parameters are \(B = 0.55\) and \(k = 8\).

4.3.1. The Optimal Random Assortment Mechanism
From Theorem 3, the optimal assortments have the following structure: each neighborhood $t$ is given an allowance $a_t$ of points, and each school $j$ costs $c_j$ points. Given allowance $a_t$, the assortment for the neighborhood is the union of:

- any school $j$ within the one-mile walk-zone which costs less than the allowance, $c_j < a_t$.
- The $k_t = k + \text{Walk}_t - 1$ schools outside of the walk-zone with the highest score $\sigma_{tj}$, defined as

$$\sigma_{tj} = \bar{u}_{tj} + \log(a_t - \text{Distance}_{tj} - c_j)$$

(32)

If fewer than $k_t$ schools have a positive sum within the logarithm, then include only the ones that do.

The costs $c_j$ are fixed, but the allowance $a_t$ may be random, and so the associated assortment can also be random. However, the assortment is deterministic for 63% of neighborhoods, and involves at most two values for 35% of neighborhoods.

These assortments can be interpreted intuitively as follows: The cost $c_j$ represents the opportunity cost of removing one unit of capacity for other students. This is higher for schools that are more popular but have lower capacities. The allowance $a_t$ represents how much “affirmative action” neighborhood $t$ gets in the allocation, with neighborhoods with lower expected utility given a higher value of $a_t$, since the minimum utility of neighborhoods is in the objective. Figure 2a in Appendix B plots the distribution of expected allowance across the city and Figure 2b plots the school costs. For schools within the one-mile walk-zone, there is no need of limiting options for the bus coverage constraint, so such schools are included as long as they are not too highly demanded, $c_j < a_t$. For schools outside of the walk-zone, we limit to $k_t = k + \text{Walk}_t - 1$ schools so that including the default school (which is modeled as the outside option as in Section 4.1.4), the total number of schools outside of one’s walk-zone is at most $k$. The formula for deciding which schools to include (Equation 32) favors schools that the neighborhood likes on average (high $\bar{u}_{tj}$), and penalizes schools that are far away or have high cost. This formula optimally balances the utility of neighborhood $t$ with externalities of others.

---

Note that the default school, being modeled as the outside option, is not technically in the assortment, but these schools are always offered to students and students can always attend them if they wish.
4.3.2. The Optimized Menus and Priority Distributions

From these optimal assortment probabilities \( \{x_tS\} \), we obtain corresponding menus, priority distributions, and quotas. For the menus, let the assortments with positive probability for each type be \( C_t = \{S : x_tS > 0\} \), we define the menu to be \( M_t = \bigcup S C_tS \) as in Equation 9.

As mentioned before, because of institutional constraints from 2012-2013 reform, we use the quotas \( \bar{q}_j \) from Section 4.1.4 instead of the optimized quotas from Equation 11. By Equation 31, the optimized quota cannot be more than the capacity limit, so this departure from theory only increases quotas at schools. By well-known properties of the DA algorithm, this yields a Pareto improvement for the students in their utilities, but may increase the busing distance.

Moreover, for the priority distributions, instead of using Equation 13 which involves the sum of two random components, we use Equation 14 which involves the sum of a neighborhood-specific deterministic term and an i.i.d. random number for each student. This is because this second form of priorities is already used in the Home-Based plan, so it is easier to adopt and explain to families. However, in order the apply Equation 14 the assortments should be nested for every neighborhood, which does not hold in our case. As a heuristic, we modify the assortments to make them nested. Let the original assortments be:

\[
C_t = \{S_1, S_2, \ldots\},
\]

with \( |S_1| \leq |S_2| \leq \cdots \) (33)

Then define new assortments \( S'_1 = S_1, S'_2 = S'_1 \cup S_2, S'_3 = S'_2 \cup S_3 \) and so on, with assortment probabilities \( x'_{tS'} = x_{tS} \). These new assortments \( C'_t = \{S'_1, S'_2, \cdots\} \) are nested by construction, so we can apply Equation 14. Let \( h_{tj} = \sum_{S' \in C'_t : S' \ni j} x'_{tS'} \), and let \( \delta_i \sim \text{Uniform}[0,1] \), then we define the priority of student \( i \) for school \( j \) to be \(^5\)

\[
\pi_{ij} = h_{tj} + \delta_i
\] (35)

We refer to the menus and priority distributions described above as the Optimized plan.

\(^5\) Note that we removed the factor of \( \frac{1}{2} \), but this is without loss of generality.
4.4. Simulation Results

We evaluate the outcomes of interest from Section 4.2 for the 3-Zone plan, the Home-Based plan, and the Optimized plan in Table 2. All the results are averages from 10,000 independent simulations according to the process described in Section 2.2, which involves sampling population, utilities, priority distributions and applying the DA algorithm. In each of the plans, the corresponding menus and priority distributions are used. The capacity limit of Section 4.1.4 is used as quotas for all three plans.

<table>
<thead>
<tr>
<th></th>
<th>3-Zone</th>
<th>Home-Based</th>
<th>Optimized</th>
</tr>
</thead>
<tbody>
<tr>
<td>Av. # of choices</td>
<td>27.5</td>
<td>14.8</td>
<td>15.0</td>
</tr>
<tr>
<td>Av. busing distance</td>
<td>1.3</td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>Av. bus coverage</td>
<td>20.9</td>
<td>8.5</td>
<td>8.3</td>
</tr>
</tbody>
</table>

Table 2 Simulation Comparisons of 3-Zone, Home-Based, and Optimized plans.

Comparing first the 3-Zone plan and the Home-Based plan, we see that the Home-Based plan gives students a smaller sized menu on average (15 as opposed to 28), and uses much less busing. Both the average distance and the bus coverage area are less than half of that in the 3-Zone plan. However, in obtaining these savings, the Home-Based plan sacrifices the expected utilities of students, both on average and for the neighborhoods with lowest utilities, and this is driven by the large decrease in the variety of choice. However, this decrease is arguably small. As a ballpark comparison, we translate this utility decrease in terms of distance, as the distance coefficient in the utilities is one. This welfare decrease is equivalent to having students travel 0.5 miles longer to every school. In terms of predictability, the Home-Based plan performs similarly to the 3-Zone plan, as measured by giving students high chances to the schools they rank first. In terms of community cohesion, students from the median neighborhood have 2 more neighbors assigned to their...
school under the Home-Based Plan. This is a sizable increase because the baseline number of neighbors in the 3-Zone plan is about 3.

Comparing the Optimized plan with the other two, we see that it attains similar savings in busing distance and bus coverage as the Home-Based plan, which is by construction. However, it does not sacrifice expected utility of students compared to the 3-Zone plan, even though the average number of schools in menu is decreased as in the Home-Based plan. It achieves this win-win situation by better optimizing the set of schools each neighborhood has access to and better accounting for the externalities on others of giving one neighborhood access to a school. (See Section 4.3.1 for intuition on how it accomplishes this balancing.)

For predictability, the Optimized plan outperforms both of the other plans. In other words, if a school appears in the menu, then the Optimized plan gives students access to the school with high probability. This is a side product of the fact that the assortment probabilities are found by solving a linear program, which produces basic feasible solutions, which naturally include a few number of non-zeros. For community cohesion, the Optimized plan achieves the gains of the Home-Based plan.

Hence, by optimization, one can produce a plan that dominates both the 3-Zone and Home-Based plans in all the outcomes of interest. The plan uses the smaller busing allowance of the Home-Based plan but gives students utilities higher than the 3-Zone plan. It also achieves better results for predictability and community cohesion, although it did not explicitly optimize for these moments.

Since the optimization involves heuristics, it may not be completely optimal. In Appendix D, we upper-bound the possible performance of any plan that uses at most .6 miles of busing per student as in the Home-Based plan, and we show that the average utility is at most 7.8. Note that the Optimized plan achieves 7.5 and also attains a small bus coverage area.

In Appendix C, we evaluate the robustness of these findings under changes in the population and utility distributions. We show that without re-optimizing, most of these gains of the Optimized plan are preserved. This shows that optimization yields gains even if the inputs are not completely correct.
5. Conclusion

We show that the menus and priority distributions in school choice systems can be systematically optimized to induce a desirable outcome, which for the Boston case study entails giving students better chances to go to the schools they want while decreasing busing burdens. The plan also improves the system’s predictability and help local communities stay cohesive.

The methodology is based on a new connection between matching and assortment planning (Theorems 1 and 2). This allows us to approximate the optimal school choice problem as a convex program, which can be efficiently solved by iteratively solving a deterministic assortment planning problem for each type of students. This subproblem is analogous to the revenue-maximizing assortment planning problem from the revenue management literature, except that the objective is social welfare rather than revenue, and we adapt algorithms from the literature to efficiently find socially-optimal assortments (Theorem 3).

To apply the optimization methodology in practice, one would estimate the population and utility distributions from past years’ data, and compute optimal menus and priority distributions for the next few years. (Since only the preferences of past years’ students enter into the optimization, the system remains strategyproof.) One would not re-optimize the menus and priority distributions every year, but only do so every 5-10 years.

References


6 Having ever changing menus is costly for busing, because each neighborhood needs a bus not only from the schools in the current menu, but in the schools in the past menus, since those assigned in previous years and have not graduated yet also need transportation.


Ashlagi, I., A. Nikzad, A. Romm. 2015. Assigning more students to their top choices: A tiebreaking rule comparison. *Available at SSRN 2585367*.


**Appendix A: The Home Based plan**

The Home-Based plan implemented in Boston in 2014 is precisely defined as follows. The menus are based on a partition of schools into four tiers based on standardized test-scores, with Tier 1 being the best. The menu of each regular education student is the union of:

- any school within 1 mile straight line distance;
- the closest 2 Tier 1 schools;
- the closest 4 Tier 1 or 2 schools;
- the closest 6 Tier 1, 2 or 3 schools;
- the closest school with Advanced Work Class (AWC);
- the closest Early Learning Center (ELC);\(^7\)
- the 3 closest capacity schools\(^8\)
- the 3 *city-wide schools*, which are available to everyone in the city.

Furthermore, for students living in parts of Roxbury, Dorchester, and Mission Hill, their menu includes the Jackson/Mann school in Allston/Brighton.

For students without older siblings in the system or special needs, the priorities were: 1) students living in East Boston had priority for East Boston schools; students outside of East Boston had priority for non-East Boston schools. 2) to break ties, each student \(i\) is given an i.i.d. random number \(\delta_i\).

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\(^7\)ELCs are extended-day kindergartens.

\(^8\)Capacity schools are those which BPS has committed to expanding capacity as needed to accommodate all students. In the 2014 implementation of the Home Based Plan, for elementary schools, the capacity schools are exactly the Tier 4 schools.
Appendix B: Additional Tables and Figures

Table 3 shows the forecasted proportion of students applying from each neighborhood. Figures 1a and 1b give a big picture view of the distribution of supply and demand for schools and of inferred school quality in Boston.

Table 3  Means and standard deviations of the proportion of K2 applicants from each neighborhood, estimated using 4 years of historical data.

<table>
<thead>
<tr>
<th>Neighborhood</th>
<th>Mean</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Allston-Brighton</td>
<td>0.0477</td>
<td>0.0018</td>
</tr>
<tr>
<td>Charlestown</td>
<td>0.0324</td>
<td>0.0024</td>
</tr>
<tr>
<td>Downtown</td>
<td>0.0318</td>
<td>0.0039</td>
</tr>
<tr>
<td>East Boston</td>
<td>0.1335</td>
<td>0.0076</td>
</tr>
<tr>
<td>Hyde Park</td>
<td>0.0588</td>
<td>0.0022</td>
</tr>
<tr>
<td>Jamaica Plain</td>
<td>0.0570</td>
<td>0.0023</td>
</tr>
<tr>
<td>Mattapan</td>
<td>0.0759</td>
<td>0.0025</td>
</tr>
<tr>
<td>North Dorchester</td>
<td>0.0522</td>
<td>0.0047</td>
</tr>
<tr>
<td>Roslindale</td>
<td>0.0771</td>
<td>0.0048</td>
</tr>
<tr>
<td>Roxbury</td>
<td>0.1493</td>
<td>0.0096</td>
</tr>
<tr>
<td>South Boston</td>
<td>0.0351</td>
<td>0.0014</td>
</tr>
<tr>
<td>South Dorchester</td>
<td>0.1379</td>
<td>0.0065</td>
</tr>
<tr>
<td>South End</td>
<td>0.0475</td>
<td>0.0022</td>
</tr>
<tr>
<td>West Roxbury</td>
<td>0.0638</td>
<td>0.0040</td>
</tr>
</tbody>
</table>

Appendix C: Robustness of the Optimization to Errors in Parameters

The optimization in Section 4 depends on distributional assumptions on the student population and preferences. In this section, we evaluate the robustness of the Optimized plan to errors in these assumptions.

The population distribution in Section 4.1 is based on data from 2010-2013, and the utility distribution is estimated from students’ submitted preferences from 2013. In this section, we re-evaluate the various plans by using the real population data from 2014 and by using a utility distribution estimated from 2014 preferences. The amount of perturbation in parameters from this computational experiment represent the typical perturbation one may observe from year to year.

The changes in the distributional assumptions are significant. For the population, instead of a forecasted total of 4294 students, only 3964 students applied in 2014. This difference is about 3 times the standard deviation of 115 students in the original population distribution. For the utility distribution, the inferred qualities of schools shifted, with the average shift being 0.69. (Recall that in the utility distribution, magnitudes are normalized to distance, so this is equivalent to changing students’ travel distances to a school by ±0.69 miles.) The estimated scale of the Gumbel distribution β changed from 1.88 to 1.64, and the estimated effect of coefficient for the walk-zone term γ changed from 0.86 to 0.37.

The simulation results using these updated parameters are in Table 4. We do not display the average number of choices and the bus coverage area since these are the same as before. Note that
Figure 1 The diagram on the left shows the distribution of students and the capacities of schools. Each blue circle represents a neighborhood, with its area proportional to the expected number of students from that neighborhood. Each yellow circle represents a school, with its area proportional to the number of K2 seats available. The capacity schools are shaded. The distribution of students is based on the 2010-2013 average. The capacities are based on data from 2013. The right shows estimates of $Q_s$ (inferred quality) from the 2013 data. The size of the circle is proportional to the estimated $Q_s$, with higher quality schools having larger circles.

although the parameters for evaluation changed, the Optimized plan is based on parameters from before, and has not been re-optimized.

We find that the optimized plan still dominates the 3-Zone and Home-Based plans in busing savings, average expected utility of students, predictability and community cohesion, despite not having the right distributional assumptions as inputs. Moreover, the magnitude of its improvement over Home-Based in these moments are similar to before.

However, the Optimized plan no longer dominates the other two in measure of equity: the 10th percentile of expected utilities of neighborhoods and the minimum expected utility. Figures 3a and 3b compare the expected utility for each neighborhood from the original simulations and the
Figure 2  These plots show the parameters that define the optimal assortments from Section 4.3.1.

Each neighborhood is giving an allowance of points, and each school costs a certain number of points. The diagram on the left shows the distribution of expected allowance $E[a_t]$ across neighborhoods, with each circle representing a neighborhood and the size of the circle proportional to the expected allowance. The diagram on the right plots the schools as circles, with the size of each circle proportional to the school costs. These are not monetary costs, but shadow costs from the optimization (OptAssortBoston) from Section 4.3.

Simulates with the updated parameters, and we find that much of the decrease is in the Hyde Park region of Boston, which is shown using a red oval. To understand what happened, we compare the school qualities from the original and updated utility distributions in Figures 4a and 4b, and we find that the inferred qualities in 2014 of schools in Hyde Park is much lower than the estimates from 2013. This shows that the equity performance of the Optimized Plan is delicate, and can be easily disturbed if school qualities in a region change in a systematic way. This makes sense because these equity measures, of 10th percentile and minimum, are susceptible to outliers. We leave the problem of finding an assignment plans that preserve equity in a way that is more robust to errors in parameters to future work.
Table 4 Re-evaluation of 3-Zone, Home-Based, and Optimized plans using updated parameters from 2014 data. All the results are averages from 10,000 independent simulations.

<table>
<thead>
<tr>
<th></th>
<th>3-Zone</th>
<th>Home-Based</th>
<th>Optimized</th>
</tr>
</thead>
<tbody>
<tr>
<td>Av. busing distance (miles)</td>
<td>1.1</td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>Expected utilities of neighborhoods</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Weighted average</td>
<td>6.7</td>
<td>6.3</td>
<td>6.7</td>
</tr>
<tr>
<td>10th perc.</td>
<td>5.8</td>
<td>5.5</td>
<td>5.7</td>
</tr>
<tr>
<td>Lowest</td>
<td>4.4</td>
<td>4.8</td>
<td>4.5</td>
</tr>
<tr>
<td>% getting top choices in menu</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Top 1</td>
<td>62%</td>
<td>61%</td>
<td>82%</td>
</tr>
<tr>
<td>Top 3</td>
<td>85%</td>
<td>83%</td>
<td>96%</td>
</tr>
<tr>
<td>Median # of neighbors co-assigned</td>
<td>3.1</td>
<td>4.5</td>
<td>4.6</td>
</tr>
</tbody>
</table>

Overall, the results are positive, as the Optimized plan preserves its gains in performance for the majority of the outcomes of interest, even under sizable errors in its distributional assumptions for population and preferences.

Appendix D: Upper-bounds on the Optimality Gap

The Optimized Plan in Section 4.3.2 involve several heuristics, so may not be completely optimal. In this section, we take advantage of the fact that (OptAssortments) is a relaxation of (OptDA) (see Section 3.1.1), and use its optimal objective to upperbound the optimal objective of (OptDA). We compute the average expected utility achievable by any plan that uses no more miles of busing per student than the Home-Based plan. To do this, we solve (OptAssortBoston) with a few modifications:

- Set a budget $B = 0.6$ for average busing. This is the average busing using in the Home-Based plan.
- Remove the MinUtility term from the objective, to only optimize the weighted average utility.
- Relax the assortment constraints to $\Psi_t = [m]$.

By the arguments in Section 3.1.1, the assortment probabilities (defined in terms of accessible assortment for each neighborhood $t$) of any plan that uses less than 0.6 miles of busing per student must be a feasible solution to this revised convex program. This is true in spite of possible randomness in the population vector, and also in spite of the fact that we are not using a large-market approximation. This convex program can be solved using the techniques in Section 3.2 and the optimal objective is 7.8. The Optimized plan from Section 4.4 achieves 7.5, despite its also optimizing minimum utility and limiting bus coverage areas. This shows that the optimality gap, at least in terms of utilitarian welfare of students, is small.
Figure 3 These plots show the expected utilities of neighborhoods under the optimized plan. Each circle represents a neighborhood and the size of the circle is proportional to the expected utility. The left plots the values under the population distribution and utility distributions from Section 4.1. The right plots the values under the actual 2014 population and re-estimated utility distribution. The red oval in the plot on the right shows the biggest area of utility decrease. This corresponds to the Hyde Park region of Boston.

Appendix E: Validation of the Large-Market Approximation for Boston

The assortment planning approximation in Section 3.1 is based on a theory requiring the number of students of each type to go to infinity. However, in the empirical exercise, the total number of students is 4294, and there are 868 types, so the average number of students per type is only 5. Nevertheless, there are reasons to expect the large-market approximation to be reasonable. Firstly, neighborhoods that are close to one another tend to have similar menus, utility distributions, and priority distributions, so there are regional pooling effects. Secondly, the independence in preferences make it so that the number of students who prefer a school from a certain area converge quickly to its expectation.
In this section, we empirically test the goodness of the large-market approximation in this data set. We do this by comparing the outcomes of interests in Section 4.2 as predicted by the large market model with the outcomes from discrete simulations.

Before showing the results, we first comment on the possible sources of discrepancies between these two types of estimates. The first possible source of discrepancy is that the market size in the Boston data is not large enough for the large-market approximation of Proposition 1 to set in. The second source is that the simulations involve randomness in the student population, while the
large-market approximation assumes the number of students of each type is fixed. The third is that
the quotas in the large-market approximation should come from Equation 11, while the simulation
in Section 4.4 use the capacity limits instead as a heuristic.

In this section, we focus on the first issue of market size. In order to do this, we modify the
population distribution and quotas in order to remove the latter two sources of discrepancy. For
the quotas, we use the quotas from Equation 11 rounded down to an integer. For the number of
students of each type $n_t$, we set it to be as close to the original mean $E[n_t]$. Precisely speaking, let
$\mu_t = E[n_t]$, we redefine a modified population distribution so that

$$n'_t = \begin{cases} 
\mu_t + 1 & \text{with probability } \mu_t - \lfloor \mu_t \rfloor, \\
\mu_t & \text{otherwise.}
\end{cases}$$

Table 5 tabulates the simulation results of the Optimized plan from Section 4.3.2 using the
modified population distribution and quotas, and compare with the predictions from the large-
market model. As can be seen, the estimates are all very similar, with the largest discrepancy coming
from the minimum expected utility of neighborhoods. This makes sense because the minimum is
equivalent to the $\frac{100}{999}$th percentile, which has a larger variance and requires a larger sample to
converge to the mean. For all of the other moments, the simulation results are within 3% of the large
market estimate. This shows that the market size in Boston is large enough for the large-market
approximation to be adequate.

| Table 5 | Comparing the predictions from the large-market model of the Optimized plan and from
simulations. The simulation results are different than those in Section 4.4 because we use a different set
of quotas and a population distribution with less variation, in order to focus on market size aspect of
the approximation. We also report a reduced set of metrics because certain metrics are difficult to
evaluate in the large-market setting without simulations. |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Av. busing distance (miles)</td>
<td>Large-market Simulations</td>
</tr>
<tr>
<td>Expected utilities of neighborhoods</td>
<td></td>
</tr>
<tr>
<td>Weighted average</td>
<td>7.62</td>
</tr>
<tr>
<td>10th perc.</td>
<td>6.79</td>
</tr>
<tr>
<td>Lowest</td>
<td>6.79</td>
</tr>
<tr>
<td>% getting top choices in menu</td>
<td></td>
</tr>
<tr>
<td>Top 1</td>
<td>89%</td>
</tr>
</tbody>
</table>
Appendix F: Omitted Proofs

Proof of Lemma 7 For convenience, denote the supremum on the right hand side of Equation 24 as RHS, and the optimal objective on the left as LHS.

First, we show that RHS ≤ LHS. This is because for any \( \lambda_0 \in \mathbb{R} \), let \( (x_0, y_0) \in A(\lambda_0) \), then \((x_0, y_0) \in \{(x, y) \in R : y = f(\lambda_0) + \lambda_0 x\}\). This implies that

\[
h(\lambda_0, f(\lambda_0)) \leq g(x_0, y_0) \leq \text{LHS}.
\]

Conversely, we show that LHS ≤ RHS. If \((x^*, y^*)\) is an optimal solution for the LHS, with optimal objective \( c = \text{LHS} \). Consider the lower contour set

\[
B = \{(x, y) \in R : g(x, y) \leq c\}
\]

By the optimality of \( c \), we have \( D \subseteq B \). Since \( g \) is quasi-convex and \( R \) is convex, \( B \) is a convex subset of \( \mathbb{R}^2 \). Since \( g \) is increasing in \( y \), \((x^*, y^*)\) cannot be in the interior of \( B \), but must lie on its boundary. By duality of convex sets, there exists an outward pointing normal of \( B \) at \((x^*, y^*)\) with direction \((-\lambda_0, 1)\). (The \( y \)-coordinate is 1 without loss of generality because \( g \) is strictly increasing in \( y \).) Let \( f_0 = y^* - \lambda_0 x^* \), then we have that \( B \), and also \( D \) is contained in the half-plane:

\[
\{(x, y) : y - \lambda_0 x \leq f_0\}.
\]

We now show that \( h(\lambda_0, f(\lambda_0)) = g(x^*, y^*) \), from which it would follow that LHS ≤ RHS. First, note that the above implies that \( f(\lambda_0) = f_0 \), so \((x^*, y^*) \in A(\lambda_0)\). Second, since \((x^*, y^*) \in \{(x, y) \in R : y = f(\lambda_0) + \lambda_0 x\}\), we have by the definition of \( h \) that \( h(\lambda_0, f(\lambda_0)) \leq g(x^*, y^*) \). Now, suppose on the contrary that \( h(\lambda_0, f(\lambda_0)) < g(x^*, y^*) \), then there must exist \((x_0, y_0) \in R\) such that \( g(x_0, y_0) < c \) and \( y_0 - \lambda_0 x_0 = f(\lambda_0) \). Since \( R \) is open and \( g \) is continuous and increasing in \( y \), there exists a sufficiently small \( \epsilon_0 \), such that if \( y_1 = y_0 + \epsilon \), then \((x_0, y_1) \in R\), \( g(x_0, y_1) < c \) and \( y_1 - \lambda_0 x_0 > f_0 \). Therefore,

\[
(x_0, y_1) \in B \text{ but } y_1 - \lambda_0 x > f_0,
\]

which is a contradiction because \( B \) is contained in the half-plane specified by Equation 36. Therefore, \( h(\lambda_0, f(\lambda_0)) = g(x^*, y^*) \), as desired.

This shows that LHS = RHS. By the above construction for \( \lambda_0 \) from optimal \((x^*, y^*)\), we get that \( \lambda_0 \) is an optimal solution to the RHS, with \((x^*, y^*) \in A(\lambda_0)\). Finally, for any optimal solution \( \lambda^* \) for the RHS, for any \((x_0, y_0) \in A(\lambda^*)\), the argument in the first paragraph shows that \((x_0, y_0)\) is also an optimal solution to the LHS. Q.E.D.
Appendix G: Socially-Optimal Assortment Planning for Other Utility Distributions

Given a set $[m] = \{1, \cdots, m\}$ of options and an outside option 0, a utility distribution $F$, a constraint set $\Psi \subseteq 2^{[m]}$ of allowable assortments, and an externalities $r_j$ for every option $j \in [m] \cup \{0\}$. The socially-optimal assortment planning problem is to find an allowable assortment that maximizes social welfare, which is defined as a weighted sum of the expected utilities of agents and the expected externalities:

$$\max_{S \in \Psi} \alpha V(S) + \sum_{j \in [m] \cup \{0\}} P(j, S) r_j,$$

where $\alpha > 0$ is the weight of the expected utilities term in the objective. Moreover, $V(S)$ is the value of assortment $S$ for agents and $P(j, S)$ is the probability of choosing option $j$ given assortment $S$. These are induced by the utility distribution $F$ as follows: if $\bar{u} \sim F$, then

$$V(S) = E[\max_{j \in S \cup \{0\}} u_j],$$

$$P(j, S) = \begin{cases} P(u_j = \max_{j' \in S \cup \{0\}} u_{j'}) & \text{if } j \in S \cup \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

This generalizes the revenue-maximizing assortment planning problem, which has $\alpha = 0$.

In section 3.2.1, we propose an algorithm for the above problem with a MNL utility model. The algorithm is polynomial-time solvable under matroid constraints $\Psi$. In this section, we study the following utility distributions:

1. The Generalized Extreme Value (GEV) model of Mcfadden (1978), which generalizes MNL and nested logit models.

The extensions to the latter two models are necessary because the original model only defines choice probabilities, but not the assortment valuations.

We give polynomial-time algorithms for the model nested logit under cardinality constraints. For the unconstrained case, we give efficient algorithms for the the multi-level nested logit model and the Markov chain based model.

G.1. Generalized Extreme Value (GEV) Model

G.1.1. Assortment Valuation and Choice Probabilities

In the MNL model, the utility of agent $i$ for option $j \in [m] \cup \{0\}$ is distributed as

$$u_{ij} = \bar{u}_j + \epsilon_{ij},$$
where $\bar{u}_j$ is a constant and $\epsilon_{ij}$ is i.i.d. Gumbel distributed, with location parameter zero and scale parameter $\beta$. The independence in the $\epsilon_{ij}$'s gives this model the Independence of Irrelevant Alternatives (IIA) restriction, which says that regardless of the assortment $S$, as long as options $j$ and $k$ are both in the assortment, the ratio $\frac{P(j,S)}{P(k,S)}$ is constant. This severely limits the type of substitution patterns possible, as adding a new option to the assortment would always attract market shares away from options $j$ and $k$ in a proportional way.

A model that bypasses this restriction is the Generalized Extreme Value (GEV) model, which has the same form as above, except that the $\epsilon_{ij}$'s are no longer independent. In the GEV model, the errors $\epsilon_{0i}, \epsilon_{1i}, \ldots, \epsilon_{mi}$ are jointly distributed according to the following CDF,

$$F(\epsilon_{ij} \leq \delta_j \text{ for all } j \in [m]) = \exp(-G(e^{-\delta_0}, e^{-\delta_1}, \ldots, e^{-\delta_m})),$$

where $G: \mathbb{R}^{m+1} \to \mathbb{R}$ satisfies the following properties:

1. **Non-negativity on the positive orthant:** $G(\bar{w}) \geq 0$ for $\bar{w} \geq 0$. ($\bar{w}$ is a $(m+1)$-dimensional vector and $\bar{w} \geq 0$ means that every component is non-negative.)

2. **Homogeneous of degree $1/\beta$:** For all $\alpha \geq 0$, $G(\alpha \bar{w}) = a^{1/\beta} G(\bar{w})$.

3. **Differentiable**, with mixed partial derivatives satisfying the sign restriction: $(-1)^k \frac{\partial^k G(\bar{w})}{\partial \bar{w}_j \cdots \partial \bar{w}_k} \leq 0$.

We call such a function $G$ a **GEV generating function with scale $\beta$**. The MNL model is a special case of GEV model with generating function

$$G(\bar{w}) = \sum_{j \in [m] \cup \{0\}} w_j^{1/\beta}. \tag{40}$$

For every option $j \in [m] \cup \{0\}$, define attraction weight $w_j = e^{\bar{u}_j/\beta}$, and let

$$w_j(S) = \begin{cases} w_j & \text{if } j \in S \cup \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

Define $\bar{w}(S)$ to be the vector $(w_0(S), w_1(S), \ldots)$. This is restricting the components in $[m] \setminus S$ to be zero.

A well-known result for the GEV model is that the assortment valuations and choice probabilities are (see Mcfadden (1978) and Bierlaire et al. (2003)):

$$V(S) = \beta (G(\bar{w}(S)) + \gamma_{\text{Euler}}), \tag{41}$$

$$P(j,S) = \frac{w_j(S) \partial_j G(\bar{w}(S))}{G(\bar{w}(S))}. \tag{42}$$

where $\gamma_{\text{Euler}} = 0.577\ldots$ is the Euler constant and $\partial_j G$ is the partial derivative of $G$ in component $j \in [m] \cup \{0\}$.

---

9 The GEV model is originally proposed in Mcfadden (1978) with scale $\beta = 1$, and the form with arbitrary scale appears in Bierlaire et al. (2003).
Solution Method

Using the above formula for assignment probabilities, the expected externalities of assortment \( S \) is

\[
R(S) = \sum_{j \in S \cup \{0\}} r_j w_j(S) \frac{\partial G(\vec{w}(S))}{G(\vec{w}(S))},
\]

and the socially-optimal assortment problem becomes

\[
\max_{S \in \Psi} \alpha \beta G(\vec{w}(S)) + R(S)
\]

The following theorem generalizes Theorem 3 for the MNL model to all GEV models.

**Theorem 4.** For the GEV utility model with generating function \( G \) with scale \( \beta \), let

\[
f(\lambda) = \max_{S \in \Psi} \{G(\vec{w}(S))(R(S) - \lambda)\},
\]

\[
A(\lambda) = \arg \max_{S \in \Psi} \{G(\vec{w}(S))(R(S) - \lambda)\},
\]

\[
\Lambda^* = \arg \max_{\lambda \in \mathbb{R}} \{f(\lambda)e^{\frac{\lambda}{\beta}}\}.
\]

then an assortment \( S^* \) is an optimal solution to Equation 44 if and only if \( S^* \in A(\Lambda^*) \) for some \( \Lambda^* \in \Lambda^* \).

**Proof of Theorem 4.** The proof is almost identical to that of Theorem 3. Let \( x(S) = G(\vec{w}(S)) \),

\( y(S) = R(S)G(\vec{w}(S)) \),

\( g(x, y) = \alpha \beta \log(x) + \frac{y}{x} \).

We plug in this formula for \( g \) into Lemma 1 as in the proof of Theorem 3 and the desired result follows. Q.E.D.

Theorem 4 suggests this solution method for the socially-optimal assortment problem.

1. Find a set \( A \), which we call a set of candidate solutions, which is a subset of \( \Psi \) such that for any \( \lambda \in \mathbb{R} \), \( A(\lambda) \cup A \neq \emptyset \), where \( A(\lambda) \) is from Equation 46.

2. Find the optimal assortment among the reduced set \( A \) by enumeration.

This is an efficient algorithm if the set \( A \) can be found efficiently and if \( |A| \) is guaranteed to be small. It turns out that many algorithms for the revenue-maximizing assortment planning problem follow the exact recipe, and since the definition of \( A \) does not depend on \( \alpha \), those same algorithms can be adapted to solve the socially-optimal case. The only difference is that the enumeration in step 2 uses a different objective.

For example, Gallego and Topaloglu (2014) and Feldman and Topaloglu (2014a) study the nested logit model, which is a GEV model with a generating function constructed as follows. Let the set
of options $[m]$ be partitioned into disjoint sets indexed by $k$. $[m] = \Omega_1 \cup \Omega_2 \cdot \cdot \cdot$. Each $\Omega_k$ is called a nest.

$$G(\vec{w}) = w_0 + \sum_{k} \left( \sum_{j \in \Omega_k} w_j \right)^{\nu_k},$$

where $\nu_k \in (0, 1]$ is called the dissimilarity parameter for the $k$th nest. Section 4 of Gallego and Topaloglu (2014) describe a polynomial time algorithm to find a set of candidate solutions $\mathcal{A}$ for the nested logit problem, with a cardinality constraint at each nest. Section 4.2 of Feldman and Topaloglu (2014a) do the same for the version of the problem in which the cardinality constraint can also be across nests. The set $\mathcal{A}$ they find satisfies the conditions we need and is guaranteed to have no more than polynomially many elements (polynomial in $m$ and the number of cardinality constraints). By Theorem 4, this immediately yields polynomial-time algorithms for socially-optimal assortment planning for nested logit utilities with cardinality constraints.

Similarly, Li et al. (2015) define a generalization of the nested logit model with multiple level of nests. In their model, the nest structure is represented by a rooted tree. Each node of the tree corresponds to a subset of options. For each node $k$, let $\Omega_k$ denote the subset of options it represent. Each node $k$ can either be an internal node or a leaf node. If it is an internal node, then it has a set of children nodes, denoted by children($k$). The subset of options $\Omega_{k'}$ of the children $k' \in \text{children}(k)$ are disjoint from one another and partition $\Omega_k$, so

$$\Omega_k = \bigcup_{k' \in \text{children}(k)} \Omega_{k'}.$$  

Each internal node $k$ also has an associated dissimilarity parameter $\nu_k$. The root node is an internal node corresponding to all the options, $\Omega_{\text{root}} = [m]$.

Under this model, the GEV generating function is

$$[G(\vec{w}) = w_0 + \sum_{k \in \text{children(root)}} G_k(\vec{w}(\Omega_k)),$$

where the functions $G_k$ are recursively defined for internal nodes:

$$G_k(\vec{w}) = \left( \sum_{k' \in \text{children}(k)} G_{k'}(\vec{w}(\Omega_{k'})) \right)^{\nu_k}.$$  

For leaf nodes $k$, the function $G_k$ is a MNL generating function limited to components $\Omega_k$ and has scale parameter equal to the product of $\beta$ as well as the dissimilarity parameter of all the internal nodes between the root node and that leaf node. (See Appendix G of Li et al. (2015) for more details on the definition of $G$.)

Section 5 of Li et al. (2015) shows how to to efficiently find a set $\mathcal{A}$ of candidate solutions for this problem, and the set they find has cardinality at most $O(m^2)$. This yields a polynomial-time algorithm for socially optimal assortment planning for multi-level nested logit models.
G.2. Markov Chain Based Utility Model

G.2.1. Assortment Valuation and Choice Probabilities

The Markov chain model of preferences is proposed by Blanchet et al. (2013) as a tractable approximation to the mixed MNL model, which McFadden et al. (2000) show can approximate any random utility model to any degree accuracy. We extend this preference model in a natural way to add a measure of preference intensity.

The Markov chain based utility model we study is as follows: each agent has an initial utility $v_0$, which is an arbitrary constant. Let $\Omega = [m] \cup \{0\}$. Define a Markov chain with $m + 1$ nodes, in which each node corresponds to an option $j \in \Omega$. There are two sets of parameters in this utility model, an arrival rate $a_j \geq 0$ for each state $j \in \Omega$, which sum to one, and a transition probability $\rho_{kj}$ from each state $k \in [m]$ to each state $j \in \Omega$. Note that there are no transitions out of the outside option.

Given any assortment $S \subseteq [m]$, consider the following stochastic process: customers arrive at each state according to the arrival rates. Whenever they arrive at one of the nodes $S \cup \{0\}$, they leave the system. Otherwise, at each time step, they follow the transition probabilities to their next state, and continue in the system until they arrive at one of the states $S \cup \{0\}$. The assortment valuations and choice probabilities are defined as follows:

$$V(S) = v_0 - E[\# \text{ of time steps before leaving}]$$

$$P(j, S) = \mathbb{P}(\text{The state when they leave the system is } j)$$

G.2.2. Solution Method

We adapt the LP-based approach of Feldman and Topaloglu (2014b) to solve the socially-optimal assortment planning problem with Markov chain utilities and no constraints ($\Psi = 2^{[m]}$).

**Theorem 5.** Consider a Markov chain based utility model with arrival probabilities $\bar{a}$ and transition probabilities $\rho$. Let $(x^*, z^*)$ be an optimal basic solution to the linear program:

$$\begin{align*}
\text{Maximize:} & \quad v_0 + \sum_{j \in \Omega} r_j x_j - \alpha \sum_{j \in [m]} z_j \\
\text{subject to:} & \quad x_k + z_k = a_k + \sum_{j \in [m]} \rho_{jk} z_j \quad \forall k \in \Omega
\end{align*}$$

Let $S^* = \{j \in [m] : x^*_j > 0\}$. $S^*$ is a solution to the socially-optimal assortment planning problem for this utility model.
Proof of Theorem 5. As in Section 1 of Feldman and Topaloglu (2014b), for any assortment \( S \), let \( R(j,S) \) be the steady state rate of people leaving state \( j \). We have that

\[
P(k,S) = \begin{cases} 
  a_k + \sum_{j \in [m]} \rho_{jk} R(j,S) & \text{if } k \in S \cup \{0\} \\
  0 & \text{otherwise.}
\end{cases}
\]

\[
R(k,S) = \begin{cases} 
  a_k + \sum_{j \in [m]} \rho_{jk} R(j,S) & \text{if } k \in [m] \setminus S \\
  0 & \text{otherwise.}
\end{cases}
\]

Therefore, for any assortment \( S \), setting \( x_k = P(k,S) \) and \( z_k = R(k,S) \) yields a feasible solution to the LP. The social welfare of this assortment is \( \sum_{j \in \Omega} r_j x_j + \alpha (v_0 - \sum_{j \in [k]} z_j) \), which is exactly the objective function of the LP. So the optimal social welfare is upper-bounded by the optimal solution of the LP.

Moreover, by Lemma 1 of Feldman and Topaloglu (2014b), the polyhedron described by inequality [51] is such that for any vertex and any \( k \in \Omega \), either \( x_k = 0 \) or \( z_k = 0 \). This implies that if \( S^* \) are as defined in the theorem, then \( P(k,S^*) \) and \( R(k,S^*) \) are exactly given by \( x_k \) and \( z_k \), so we can attain the optimal social welfare with assortment \( S^* \). Q.E.D.

G.3. Non-Parametric Utility Model

G.3.1. Assortment Valuation and Choice Probabilities

The non-parametric model of preferences in Farias et al. (2013) is that agents are randomly drawn from finitely many customer segments, each of which has a deterministic preference ordering. They show that having such a flexible model may yield large gains in prediction accuracy compared to parametric models. For our setting, we endow each customer segment with a deterministic utility vector, which is necessary as we also need a measure of preference intensity.

Specifically, the model is that there are \( K \) customer segments, and for each \( k \in \{1, \cdots K\} \), there is a deterministic utility vector \( \vec{u}_k \) and a probability \( p_k \). The probabilities sum up to one over all \( K \) segments. For simplicity, we assume that none of the utility vectors \( \vec{u}_k \) has any two components being equal, so there are no ties in preferences.

The assortment valuations and choice probabilities are

\[
V(S) = \sum_{k=1}^{K} p_k \max_{j \in S \cup \{0\}} \{u_{kj}\} 
\]

\[
P(j,S) = \begin{cases} 
  \sum_{k=1}^{K} p_k \mathbb{1}(u_{kj} = \max_{j' \in S \cup \{0\}} \{u_{kj'}\}) & \text{if } j \in S \cup \{0\}, \\
  0 & \text{otherwise.}
\end{cases}
\]
G.3.2. Solution Method

Since this utility model generalizes the choice model of Farias et al. (2013) and since the socially-optimal assortment problem is a generalization of the revenue-maximizing problem, the hardness results in Aouad et al. (2015) carry through to our case, so the optimal solution is NP-hard to approximate within any factor $O(\min(m, K)^{1-\epsilon})$ for any $\epsilon > 0$.

For small number of options or number of segments, we can adapt the mixed integer program (MIP) formulation of Bertsimas and Misic (2015). There is a binary variable $x_j \in \{0, 1\}$ for whether option $j \in [m]$ should be included in the assortment, and continuous variable $y_{kj} \in [0, 1]$ for whether the $k$th segment chooses option $j \in [m] \cup \{0\}$.

\[
\text{Maximize:} \quad \sum_{k=1}^{K} \sum_{j=0}^{m} p_k y_{kj} (\alpha u_{kj} + r_j)
\]

\[
\text{Subject to:} \quad y_{kj} \leq x_j \quad k \in [K], j \in [m]
\]

\[
\sum_{j' \in [m] \cup \{0\}: u_{kj'} > u_{kj}} y_{kj'} \leq 1 - x_j \quad k \in [K], j \in [m] \cup \{0\}
\]

\[
\sum_{j=0}^{m} y_{kj} \leq 1 \quad k \in [K]
\]

\[
x_j \in \{0, 1\} \quad j \in [m]
\]

\[
y_{kj} \geq 0 \quad k \in [K], j \in [m] \cup \{0\}
\]

Bertsimas and Misic (2015) report positive computational results for this model in their computational experiments. Since the above formulation is the same as theirs except for the objective function (which adds the term $\alpha u_{kj}$), similar results should hold in this case.